Characterization of LMST Graphs

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Note that this is a draft and there are some citations missing and also some results. Comments in brackets [like this one] are from the authors to the authors to remind themselves what has to be done! Also, please do not have a look that is too close to the pictures but take them as sketches!

1 Definitions

Let $G = (V, E)$ be an embedded graph such that every vertex is assigned a position by $P : V \to \mathbb{R} \times \mathbb{R}$. Let $d_G(v, w)$ denote the geometrical distance between the position of v and w. Let r denote some constant that represents the *radius of visibility* of each vertex, then $E = \{(v, w)|d_g(v, w) \leq r\}$, i.e., every vertex is connected to all the vertices that are within its radius of visibility. Such a graph is called a *unit distance graph*. The set of all direct neighbors of a vertex v is denoted by $N_r(v)$ where the index indicates the radius of visibility to build this set.

In the following we will assume that the graph is connected. A global *minimal spanning tree* $T \subseteq E)$ is defined as a subset of edges with cardinality $n - 1$ that keeps the connectedness of the graph and minimizes the sum of distances of all the edges. A *local minimal spanning tree* is also a subset of edges that keeps the connectedness of a graph but with another objective. *Local minimal spanning trees (LMST)* are not defined by minimizing a global objective like the sum of the lengths of all edges but by minimizing a local objective, i.e, the sum of the lengths of all edges in the neighborhood of every vertex. Note that despite the naming, LMSTs are generally no trees but have cycles. LMSTs come in two flavors that differ in the construction of the chosen subset of edges:

1. $LMST^+$:

$$LMST^+(r) = \{(u, v)|(u, v) \in N(u) \vee (u, v) \in N_r(v)\} \tag{1}$$

2. $LMST^-$:

$$LMST^-(r) = \{(u, v)|(u, v) \in N(u) \wedge (u, v) \in N_r(v)\} \tag{2}$$

In the rest of this summary we will try to characterize $LMST^+$ and $LMST^-$ graphs, i.e., understand whether and how the following questions can be answered:

1. Given a graph $G = (V, E)$ is it possible to decide whether the vertices of G can be positioned such that the $LMST^{+/-}$ of this placement is G for some radius r?
2. Given a graph $G = (V, E)$ that is an $LMST^{+/-}$ is it possible to compute a placement of the vertices and a radius r such that the $LMST^{+/-}$ of this embedding under r is G?

Note that for the first problem we will use the following abbreviation *G is an LMST* to denote that *there is a placement function $P : V \to \mathbb{R} \times \mathbb{R}$ and a parameter r such that E of G is the $LMST(r)$ of the placed vertices.*

2 Results derived so far

2.1 Known Properties

The first known property is the inclusion property:

$$MST \subseteq LMST^- \subseteq LMST^+ \subseteq RNG \subseteq DTG \tag{3}$$

where RNG denotes the *relative neighborhood graph* where two vertices v, w are connected if there is no other vertex x with the following property:

$$d_G(v, w) \leq \min_{x \in V}\{d_G(v, x), d_G(w, x)\} \tag{4}$$

, i.e., there is no other vertex that is nearer to either v or w, and where DTG denotes the Delaunay-Triangulation-Graph. Note that the DTG is the dual of the Voronoi diagram.

MSTs are planar and connected iff the corresponding unit distance graph is connected. We will further assume that LMSTs do not contain any triangles. Triangles could derive from three vertices that are placed in equidistance and thus have three different MST to choose from. We will assume a tie breaking rule that avoids this constellation without specifying this rule. With any tie breaking rule, cycles can only emerge if none of the participating vertices can see all other vertices of the cycle.

2.2 Trees

Proposition: 1 *All trees with maximal degree 5 are LMSTs.*

Building on a known result on trees [citation is missing]: All trees with maximal degree 5 are embeddable such that the embedded tree is the MST of the vertex set. If a tree is embeddable as an MST than it is also an LMST if r

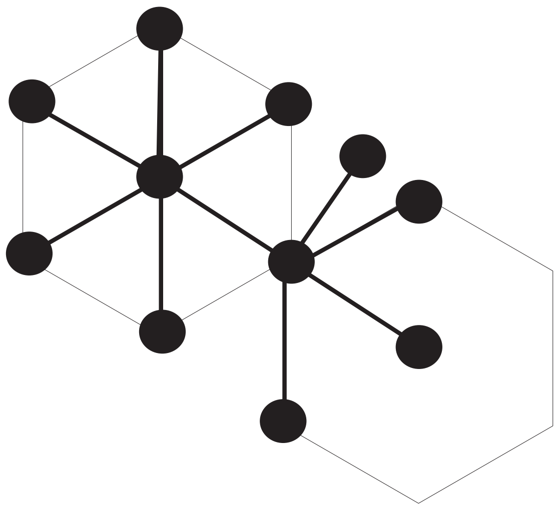

Figure 1: A tree that is not embeddable such that it is an LMST

is such that $N(v) = V$ for all the vertices $v \in V$. Thus, all trees with maximal degree 5 are LMSTs. For degree 6 it is NP-hard whether the tree is embeddable as an MST or not. Fig. 1 shows an example for a tree with degree 6 that is not an (L)MST.

Proof:
The edges incident to a vertex v with degree 6 must be of the same length l and the angle between two adjacent edges is 60. Let w be one of the neighbors of v. The circles drawn around the other neighbors of v in Fig. 1 denote forbidden zones in which none of the neighbors of w is allowed to be placed. The figure shows the Voronoi region of w by red lines. The new vertex z has to satisfy the following constraints:

$$d_G(w, z) \leq d_G(w, a) \tag{5}$$
$$d_G(v, w) + d_G(w, z) \leq d_G(a, z) + d_G(w, z) \tag{6}$$

The latter equation describes the possibility that the new MST removes the edge between v, w and replaces it by first using edge (a, z) and then (w, z). Of course, this reduces to:

$$d_G(w, z) \leq d_G(w, a) d_G(v, w) \leq d_G(a, z) \tag{7}$$

Analogous constraints are given for vertex b and z. The first constraint requires z to lie in the Voronoi region of w, the second constraint can be reformulated to:

$$l \leq d_G(a, z) \tag{8}$$

,i.e., we can draw a circle with unit length l around each, a and b, and z is not allowed to lie in these circle. The remaining area can at most contain 3 vertices

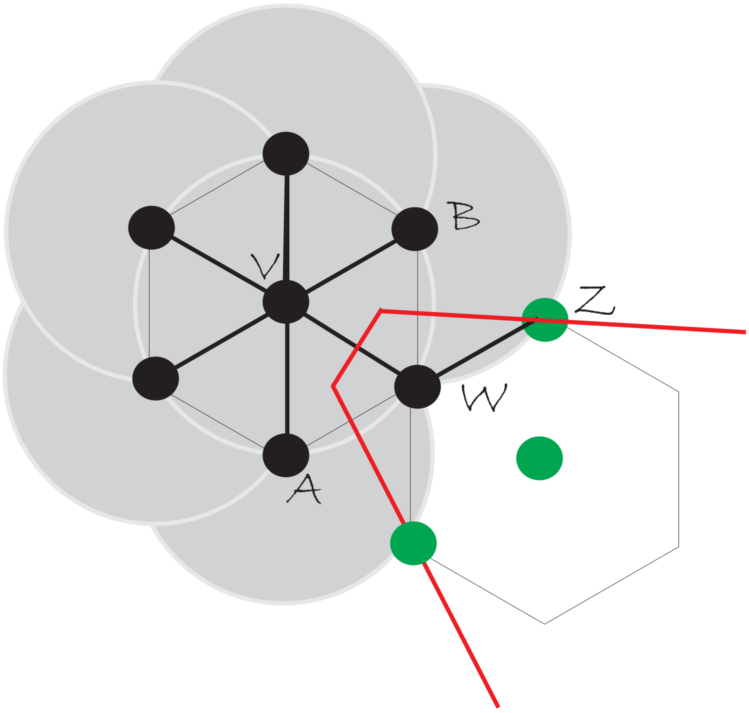

Figure 2: The vertex with degree 6 forces its neighbors to be in unit distance and adjacent edges to have an angle of 60. The red line denotes the Voronoi-Region of w before the new child z is inserted, the grey circles denote forbidden zones. The only possibility to place 3 non-connected vertices into this region is indicated by the green vertices.

that are not connected to each other and this is only possible if these vertices are positioned as it is shown in Fig. 2, where the positions are colored green.

2.3 Forbidden Subgraphs

There is a simple counterexample that shows that it is not sufficient for a graph to be planar, connected, and free of triangles to be an LMST:

Proposition: 2 *There is a planar, connected, and triangle-free graph G that is not an LMST.*

Proof: 1 *Fig. 3 shows a simple counter-example. Vertex B is surrounded by 1,c,2,a, and such the sum of the angles* $\overline{1bc}, \overline{cb2}, \overline{2ba}, \overline{ab1}$ *must be* 360*, i.e., one of the angles must be at least* 90*. Thus, one of the edges* $(1,c), (c,2), (2,a), (a,1)$ *is opposite of an angle with more than* 90 *and thus it is the longest in the*

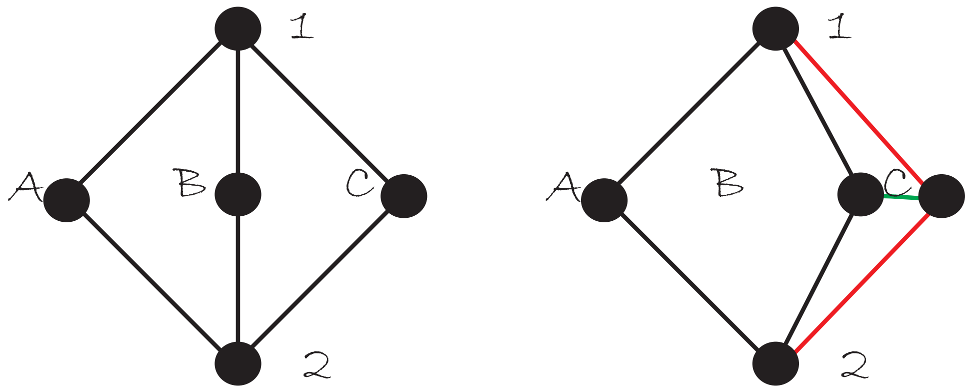

Figure 3: A simple but non-trivial graph that is not an LMST

corresponding triangle containing b. *W.l.o.g., let* $\overline{1bc}$ *be this angle, then it follows that the correct LMST of this set of vertices would neither contain edge* $(1, c)$ *nor* $(2, c)$ *but would replace it by* (b, c).

2.4 Upper Bound on the Number of Edges

A trivial upper bound on the number of edges is given by the fact that every LMST is planar, is triangle-free, and no vertex has a larger degree than 6. Thus, the trivial upper bound on the number of edges evaluates to $2n - 4$. It can be shown that there is a graph with $2n - 2\sqrt{n}$ edges. Thus, the following proposition holds:

Proposition: 3 *The maximal number* E_{max} *of edges in an LMST is bounded by* $2n - 2\sqrt{n} \leq E_{max} \leq 2n - 4$.

For outerplanar graphs the following propostion holds:

Proposition: 4 *Every outerplanar graph that is an LMST has a maximal number* E_{max} *of* $\frac{3}{2}n - 2 \leq E_{max} \leq 2n - 4$.

2.5 LMST recognition is NP-hard

Lemma: 1 *Given a graph* G *it is NP-hard to decide whether there is a placement of the vertices such that* E *is the LMST of the placed vertices.*

Proof: 2 *The proof is given by reducing an instance of* $3 - SAT$ *to the problem of deciding whether a given graph is an LMST. To do so, we have to show that there is a geometric representation of the* $3 - SAT$ *instance that can only be drawn as an LMST if the instance is satisfiable. The proof is adapted from [1],*

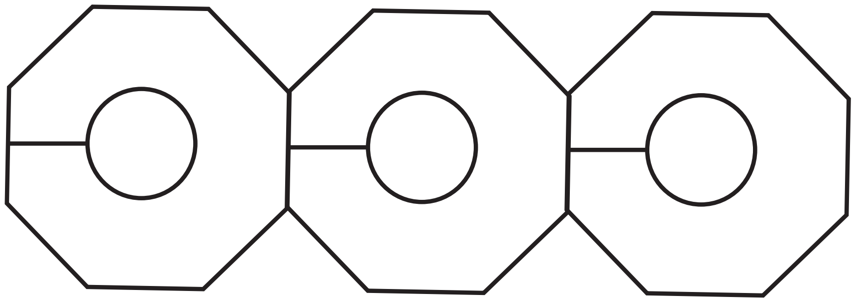

Figure 4: Cages are connected to each other such that every inner structure can be placed in either of the adjacent cages.

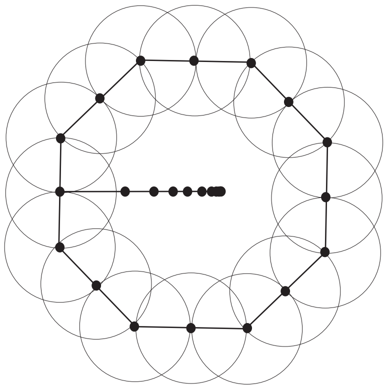

Figure 5: Within a cycle of every length, a chain composed of every wanted number of vertices can be drawn in an LMST.

so we just show the main gadget that has too be changed such that the proof from [1] is applicable. The idea behind the main gadget is to build a cage in which an inner structure can be placed. Then, these cages are connected in a chain. Due to this connection, an inner structure can be placed in either of the two adjacent cages (s. Fig. 4).

The inner structure of each cage must fill the space within the cage such that no two inner structures can be drawn within the same cage. In the case of unit distance graphs as in [1] it is enough to make this inner structure a chain. In the case of LMSTs a chain of every length can be put into a cage as long as the edge length between adjacent vertices strictly decreases (Fig. 5).

In LMST graphs we need a more rigid inner structure. As has been argued before (Sub. 2.2), a vertex with degree 6 *is putting the following constraints on its drawing: All neighbors of the vertex must be in the same distance to the vertex and adjacent edges enclose an angle of* 60*. With this the main gadget*

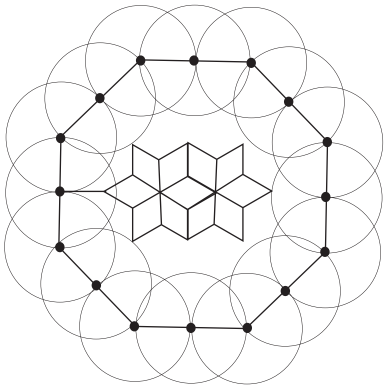

Figure 6: The outer circle builds a cage in which the inner structure can be embedded such that the edges constitute the LMST of the vertices for a radius that equals the distance of any two vertices on the outer circle.

is constructed like this: Take an outer ring of 16 *vertices. Let* l_{max} *denote the maximal length of any edge in any given drawing of this* 16 *vertices. Since we have to show that only one of the inner structures can be placed within the circle we have to consider the maximal area that can be enclosed by the circle. It is easy to show that the maximal inner area is given if all the edges have the same length of* l_{max} *and if the drawing is a regular* 16*-ogon. It is also clear that* l_{max} *is smaller or equal than the radius of visibility* r*. Thus, none of the vertices of the inner structure is allowed to be placed nearer to any of the outer vertices than distance* l_{max}*. This forbidden inner zone can be visualized by drawing circles with radius* l_{max} *around every outer vertex. The inner area has at most a radius of* $5.126 \cdot l_{max}$*. This leaves an inner area with radius* $3.126 \cdot l_{max}$ *after subtracting the forbidden zones. The inner structure is thus constructed by merging two stars as can be seen in Fig. 6.*

Now, the main gadget can be combined to form chains. There is a second gadget in which two smaller, but not three of the smaller inner structures can be placed. The proof of concept is given by Fig. 8 [Proper calculation has to be done!].

With these two gadgets the whole proof of [1] is applicable and thus the NP-hardness is proven.

2.6 $\phi - LMST^{+/-}$-Drawability

Let G be a graph with $P : V \to \mathbb{R} \times \mathbb{R}$, i.e., the vertices are placed in the plane. Let $LMST(r)$ be either of the two LMST variants under the radius of visibility r. Let l denote the minimal length of any edge in $LMST(r)$. ρ denotes the following ratio for a given $LMST(r)$ to be:

$$\rho = \frac{l}{r} \tag{9}$$

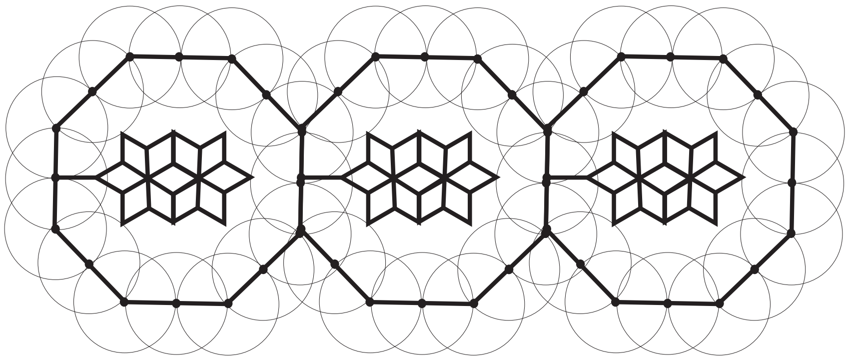

Figure 7: The main gadget can be combined to form chains. These chains ensure that no two inner structures can be in the same cage.

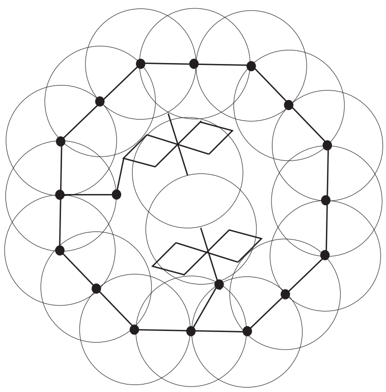

Figure 8: An inner structure that only fills half of the inner area such that exactly one other half inner structure can be placed into the same cage but not more. The circles round the vertices next to each other show that they are indeed out of each other's radius of visibility.

Note that an edge of any length l' can only exist if r is at least as great as l'. It follows directly that $\rho \leq 1$ for all LMSTs.

A graph G is ϕ-drawable if it is possible to position the vertices $v \in V$ such that E is the $LMST$ of V under some radius of visibility r with ratio ρ.

Theorem: 1 *A tree with maximal degree less than or equal to* 5 *is* $\frac{1}{2^n}$*-drawable.*

Proof: 3 *The proof is based on the construction for the MST-drawability of trees with maximal degree* 5 *(s. Sub. 2.2). [citation missing]*

Theorem: 2 *A graph that is homeomorphic to the* $n_1 \times n_2$ *orthogonal or hexagonal grid is* $\frac{1}{2(n-n_1 \cdot n_2+\epsilon)}$*-drawable, with* $\epsilon > 0$.

Proof: 4

Theorem: 3 *An outerplanargrpah with maximal degree less or equal to* 4 *whose dual is a path is* $f(n)$*-drawable.*

[f(n) still has to be computed].

Proof: 5 (proof by figure)

Theorem: 4 *Any planar graph with maximum degree* 5 *is homeomorphic to a planar graph with* $O(n^3)$ *vertices that is* 1*-drawable.*

Proof: 6

Theorem: 5 *Any planar graph* G *with maximum degree* 4 *is homeomorphic to a planar graph with* $O(n^2)$ *vertices that is* 1*-drawable.*

Proof: 7 *The proof is given by the following, constructive algorithm: Construct an orthogonal grid drawing of* G *with an area in* $O(n^2)$*. Insert vertices along the edges at grid points and at half grid points. Then, for a radius of half the unit length, the resulting graph is an LMST of the vertices. Since all edges have a length of half a unit,* ρ *is equal to* 1.

Note that the last two theorems state that there is an infinite family of planar graphs that are drawable as LMSTs.

2.7 Open Questions - to name but a few!

1. Define $N_{k,r}(v)$ to be the set of vertices w in a unit distance graph under r that are in hopping distance $d(v,w)$ up to k from vertex v:

$$N_{k,r} = \{w | d(v,w) \leq k\} \tag{10}$$

   and $LMST^+(k,r)$ to be

$$LMST^+(r) = \{(u,v) | (u,v) \in N(u) \vee (u,v) \in N_k, r(v)\} \tag{11}$$

   Let $LMST^-(k,r)$ be defined analogously.

   What properties do these family of graphs have?

2. Can we give any expected properties of an LMST with respect to the density of vertices per area (in a uniform distribution) and the radius of visibility? How much worse is the distance on the LMST compared with the distance in the full graph? What is the connectivity of the resulting graph?s

References

[1] Fabian Kuhn, T. Moscibroda, and Roger Wattenhofer. Unit disk graph approximation. In *Workshop on Discrete Algorithms and Methods for Mobile Computing and Communications*, 2004.

2.8 Acknowledgement

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