Good Drawings and Rotation systems of Complete Graphs

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GRAPH DRAWING 2014

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"Recent Trends in Graph Drawing
Curves, Crossings, and Constraints"

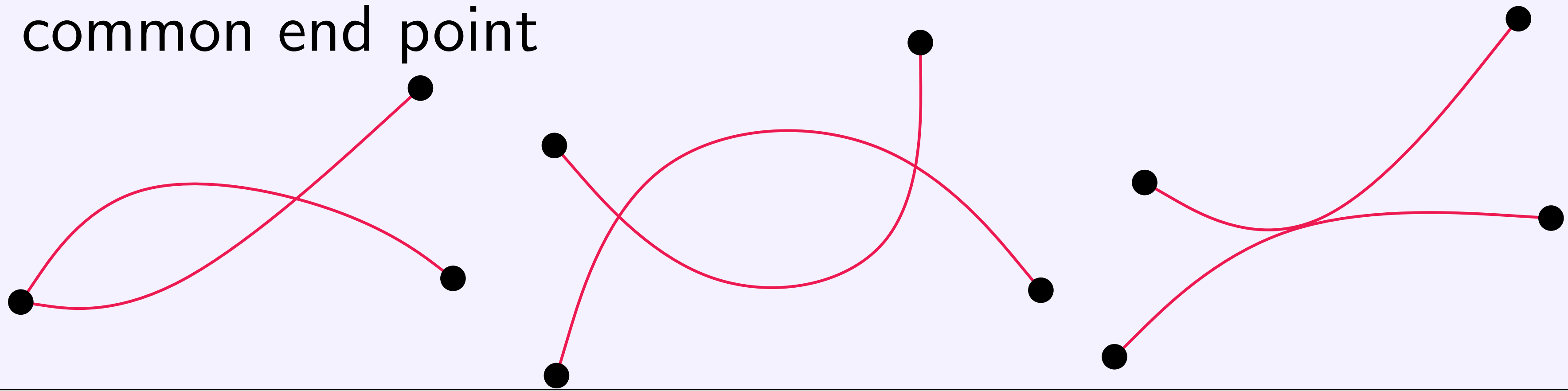
Good Drawings

Simple complete topological graph: drawing of a simple complete graph in the plane (on the sphere)

Vertices are distinct points

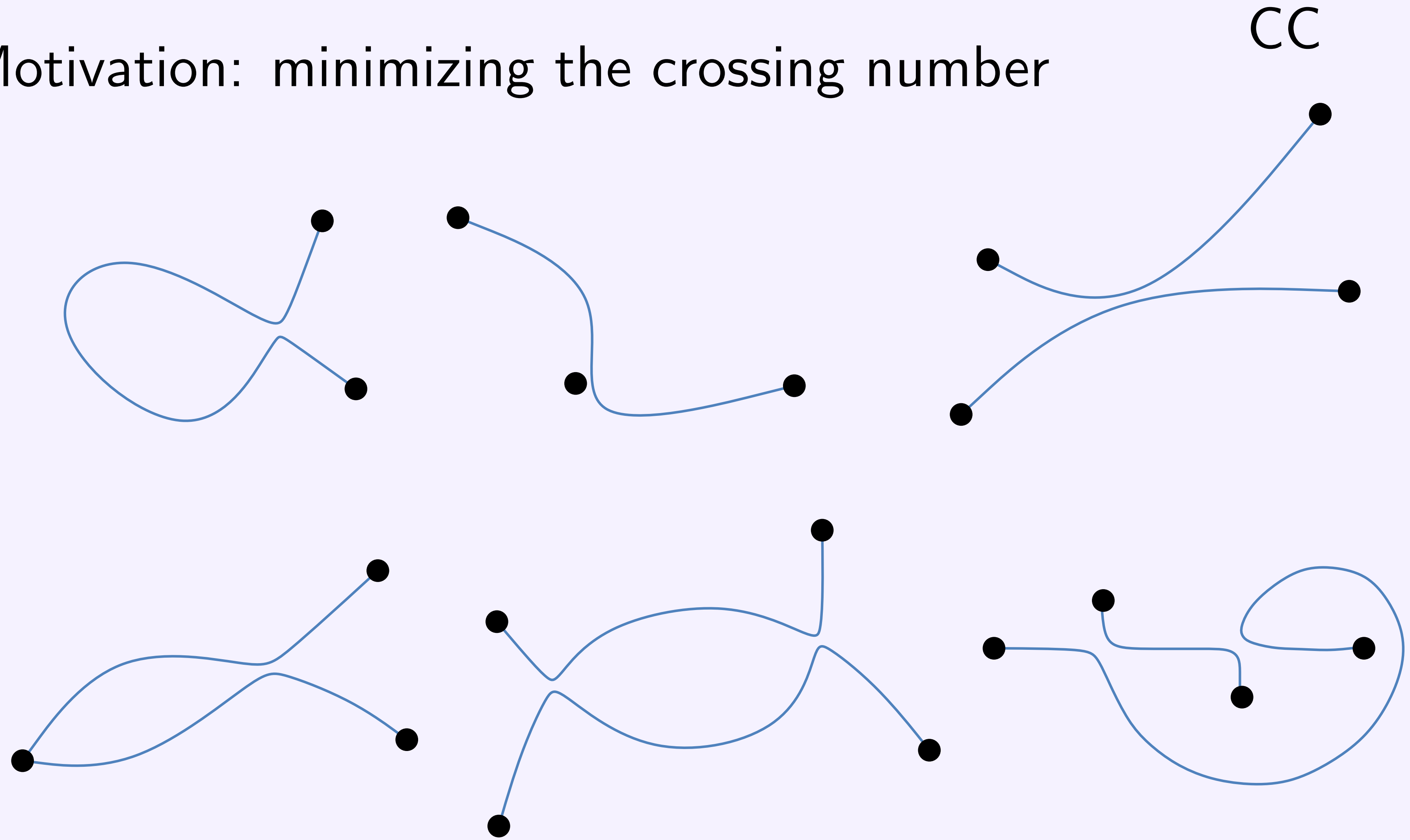
edges are non-self-intersecting continuous curves connecting two (end) points; edges do not pass through vertices

Any pair of edges intersects at most once: proper crossing or common end point

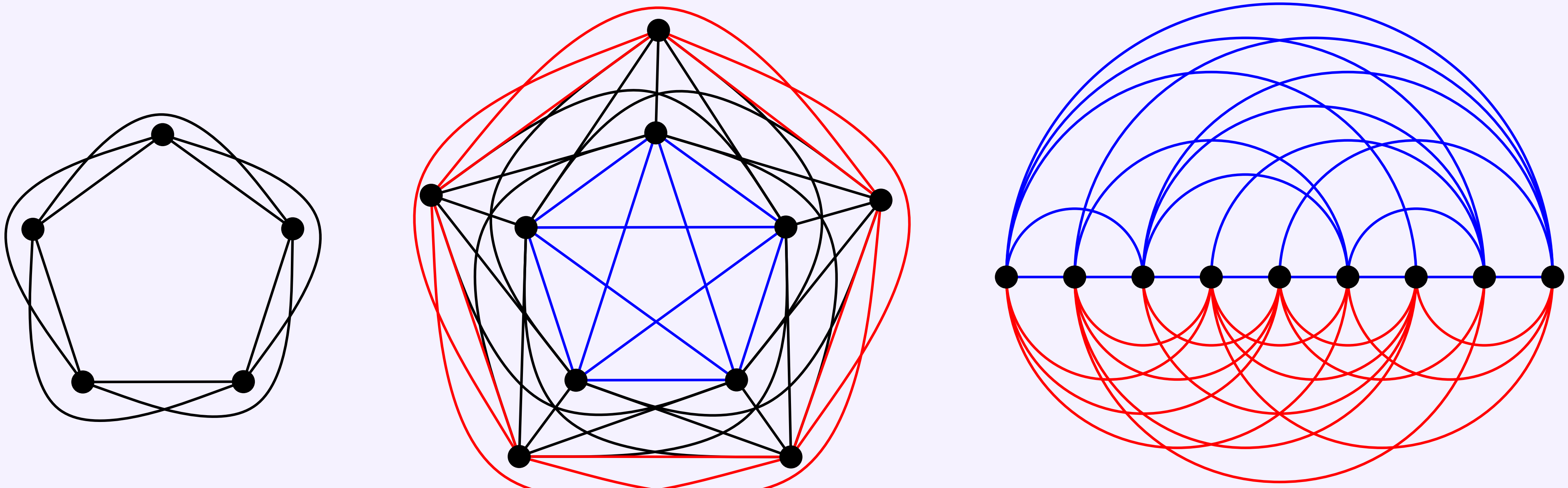


Good Drawings

Motivation: minimizing the crossing number



Some Good Drawings



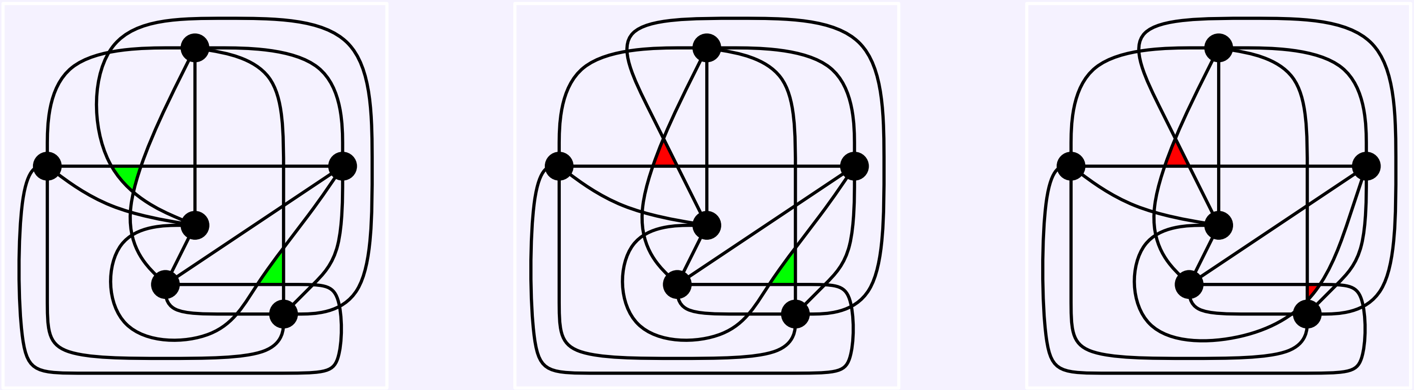
Isomorphism Classes

Two good drawings are isomorphic, if they can be obtained from each other by a homeomorphism on the sphere.

$\Rightarrow$ all vertex-edge-face incidences are the same

The number of isomorphism classes of good drawings of K_n is $2^{\Theta(n^4)}$ [Kynčl 2009]

Weakly isomorphic:



Weakly Isomorphism Classes

Two good drawings of K_n are weakly isomorphic if the same set of pairs of edges cross.

$T_w(K_n)$ number of weakly isomorphism classes of good drawings.

$2^{\Omega(n^2)} \leq T_w(K_n) \leq ((n-2)!)^n = 2^{\mathcal{O}(n^2 \log n)}$
for geometric graphs: $2^{\Theta(n \log n)}$ [Pach, Tóth 2004]

$T_w(K_n) \leq 2^{n^2 \alpha(n)^{O(1)}}$
$\alpha(n)$ is the inverse Ackermann function [Kynčl 2013]

Rotation System

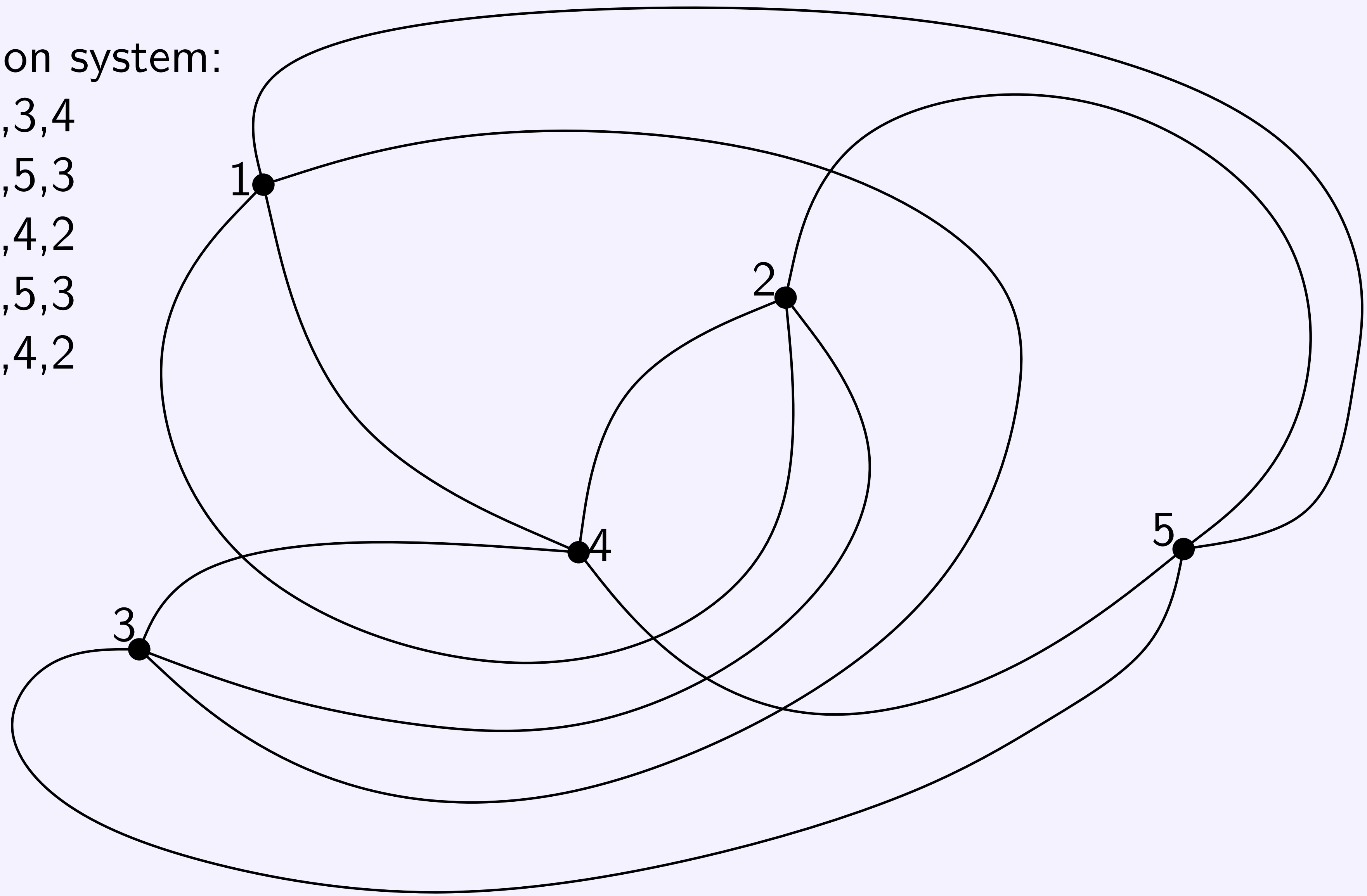
A rotation system of a good drawing of a complete graph gives for each vertex v of the graph the clockwise circular ordering around v of all edges incident to v.

[Heffter 1891; used for embedding graphs in orientable surfaces]

CCC

Peepholes

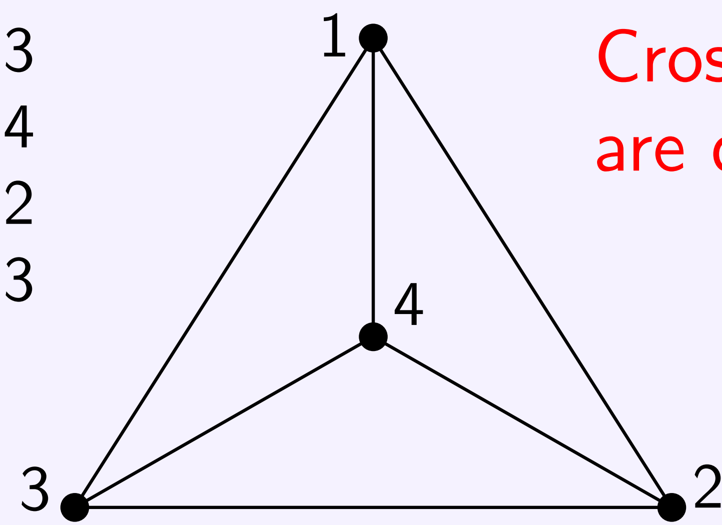
Rotation system:
1: 2,5,3,4
2: 1,4,5,3
3: 1,5,4,2
4: 1,2,5,3
5: 1,3,4,2



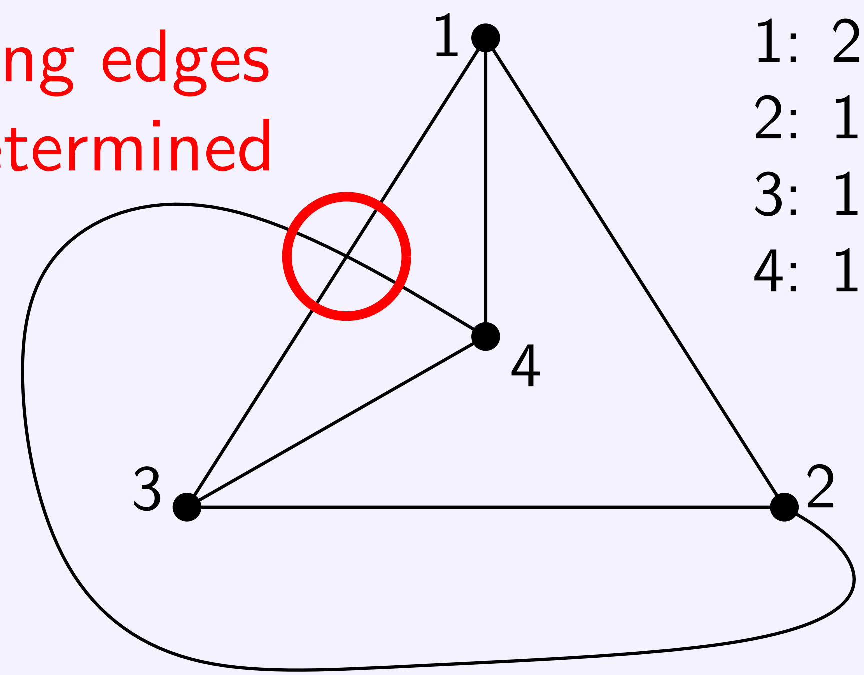
Rotation System

A rotation system of a good drawing of a complete graph gives for each vertex v of the graph the clockwise circular ordering around v of all edges incident to v.

1: 2,4,3
2: 1,3,4
3: 1,4,2
4: 1,2,3



Crossing edges are determined



1: 2,4,3
2: 1,4,3
3: 1,4,2
4: 1,3,2

For $n = 4$ there are two different (realizable) rotation systems, that is, 2 non-isomorphic good drawings

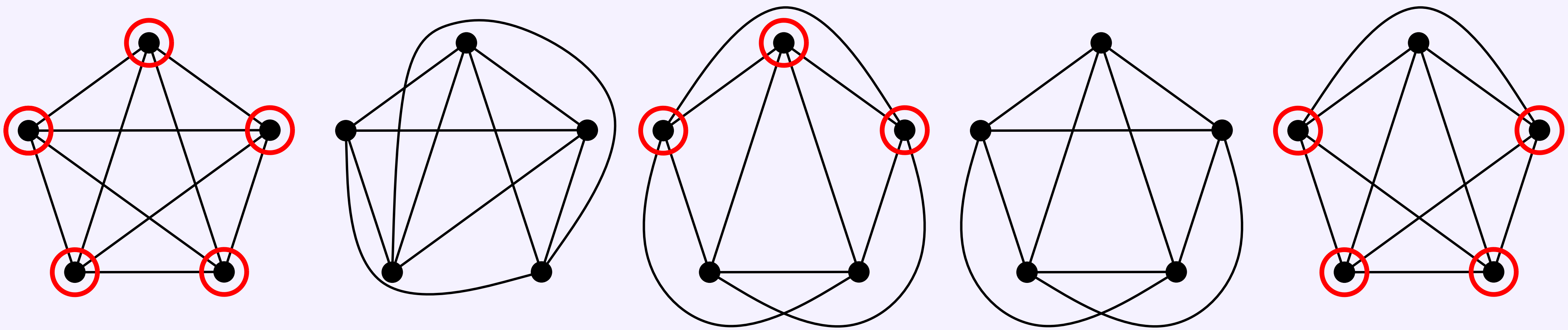
Isomorphism Classes

Two good drawings of K_n are weakly isomorphic if they have the same (or inverted) rotation system. [Kynčl 2009]

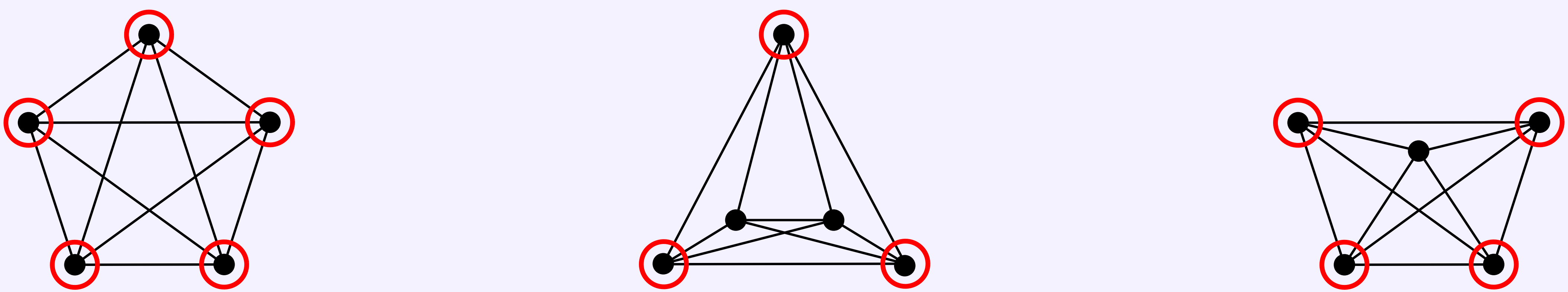
Two good drawings are isomorphic if they
(1) are weakly isomorphic
(2) the order of crossings along an edge is the same
(3) for each crossing the rotation of edges is the same (or inverted) [Kynčl 2009]

Given the crossing pairs of K_n it can be decided in polynomial time whether the graph can be realized as a good drawing [Kynčl 2011]

Rotation Systems for $n = 5$



$n = 5$: 5 different rotation systems = 5 non-isomorphic good drawings



3 different geometric drawings = 3 order types

Rotation Systems for $n \leq 9$

n	realizable rotation systems	non-isomorphic drawings	non-isomorph. drawings per rot. sys.	order types
3	1	1	1...1	1
4	2	2	1...1	2
5	5	5	1...1	3
6	102	121	1...3	16
7	11 556	46 999	1...57	135
8	5 370 725	502 090 394	1...46 571	3 315
9	7 198 391 729	?	?	158 817

For $n = 6$ there are 121 non-isomorphic good drawings
[(Mengerson 1973: 123) Gronau,Harborth 1990: 121]
Number of RS for $n = 9$ to be verified.

Geometric Crossing Number: The last 10 years

Minimal number of crossings in geometric drawing of K_n:

Relation to k-edges and halving lines:

$$\overline{cr}\,(D) = 3\binom{n}{4} - \sum_{k=0}^{\lfloor n/2 \rfloor - 1} k\,(n-2-k)\,E_k\,(D)$$

E_k ... number of k-edges [Ábrego, Fernández-Merchant; Lovász, Vesztergombi, Wagner, Welzl; 2004/2005]

Structural result: Outer onion layers are triangles [A.,García, Orden, Ramos 2007]

Exact values for all $n \leq 27$ and $n = 30$

Improved bounds for the geometric crossing constant

$q_* = lim_{n \to \inf} \frac{\overline{cr}(K_n)}{\binom{n}{4}}$: $\quad 0.379972 < q_* < 0.380473$

Crossing Number for Good Drawings

Harary-Hill Conjecture (ca. 1958):
$cr(n) \geq Z(n) := \frac{1}{4} \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor$

1972 [Guy]: exact crossing numbers for $n \leq 10$
2007 [Pan, Richter]: $cr(11) = 100$ and $cr(12) = 150$

Bruce Richter, Banff workshop on crossing numbers 2011:

“For good drawings there exists the Harary-Hill conjecture, but not much progress in recent years. For the rectilinear crossing number there was tremendous progress, and they still do not even have a conjecture for it ...”

Crossing Number for Good Drawings

Number of $\leq k$-edges: $E_{\leq k}(D) := \sum_{j=0}^{k} E_j(D)$

Number of $\leq\leq k$-edges: $E_{\leq\leq k}(D) := \sum_{j=0}^{k} E_{\leq j}(D) =$

$$\sum_{j=0}^{k} \sum_{i=0}^{j} E_i(D) = \sum_{i=0}^{k} (k+1-i) E_i(D)$$

Exact crossing number for a good drawing D of K_n:

$cr(D) = 2\sum_{k=0}^{\lfloor n/2 \rfloor - 2} E_{\leq\leq k}(D) - \frac{1}{2}\binom{n}{2}\left\lfloor \frac{n-2}{2} \right\rfloor -$
$\frac{1}{2}(1+(-1)^n) E_{\leq\leq \lfloor n/2 \rfloor - 2}(D)$

[Ábrego, A., Fernández-Merchant, Ramos, Salazar, 2011/12].

Shellable Drawings

A drawing D of K_n is s-*shellable* if there exists a sequence $S = \{v_1, v_2, \ldots, v_s\}$ of a sub set of the vertices and a region R of D such that if for all $1 \leq i < j \leq s$ holds: D_{ij} is the drawing obtained from D by removing $v_1, v_2, \ldots v_{i-1}, v_{j+1}, \ldots, v_s$, then v_i and v_j are on the boundary of the region of D_{ij} that contains R.

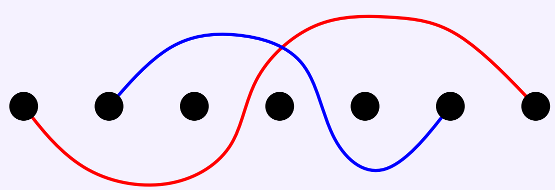
For $s \geq \lfloor n/2 \rfloor$ and any s-shellable drawing D of K_n:
$cr(D) \geq Z(n) := \frac{1}{4} \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor$

[Ábrego, A., Fernández-Merchant, Ramos, Salazar, 2013]

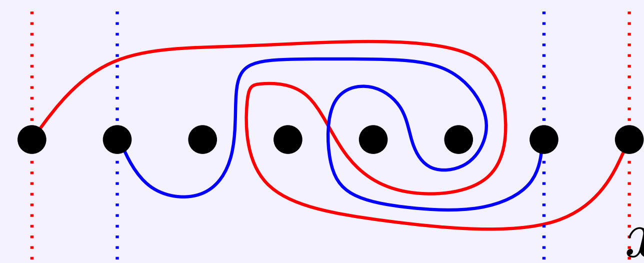
First combinatorial classification to identify drawings for which the Harary-Hill conjecture provably holds.

Shellable Drawings

Monotone, x-bounded drawings of K_n are s-shellable ($s \geq \frac{n}{2}$)

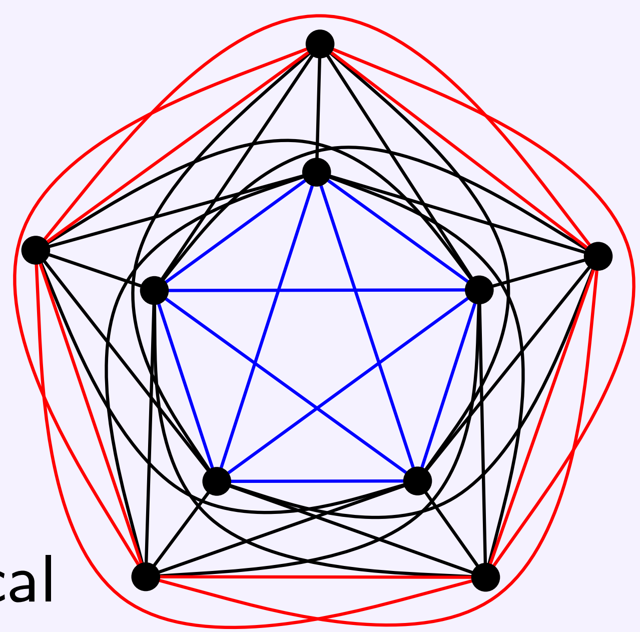


monotone

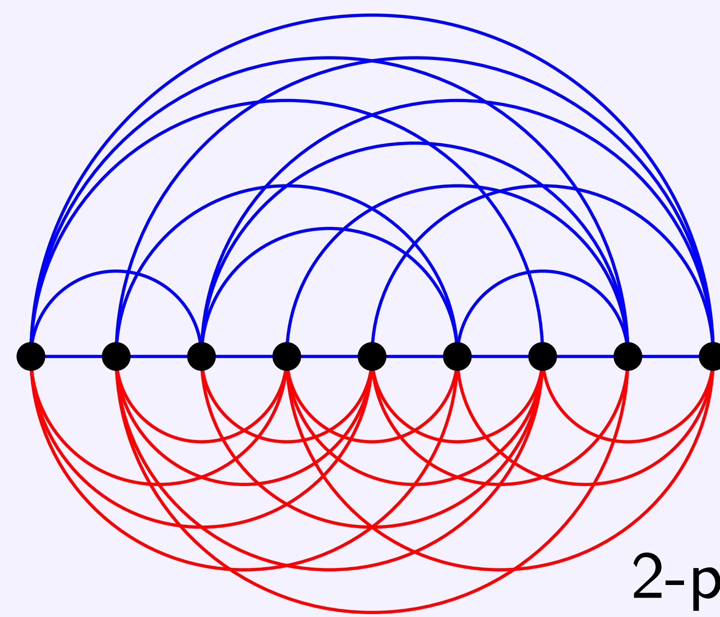


x-bounded

Cylindrical and 2-page drawings of K_n are s-shellable ($s \geq \frac{n}{2}$)

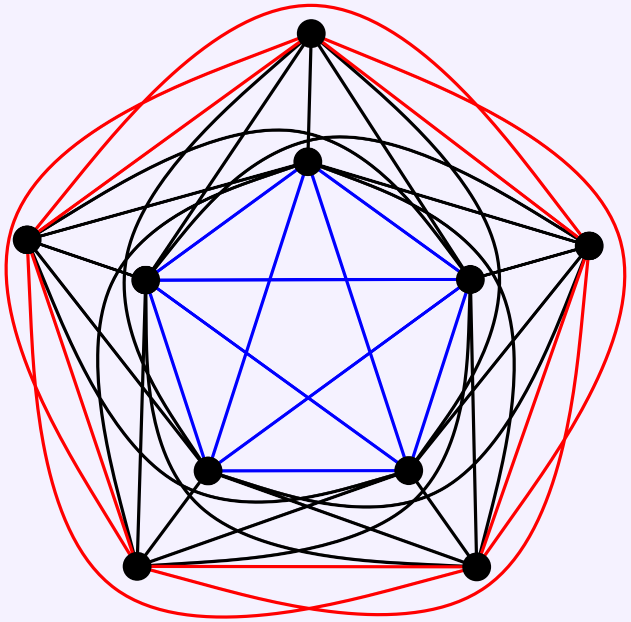


cylindrical

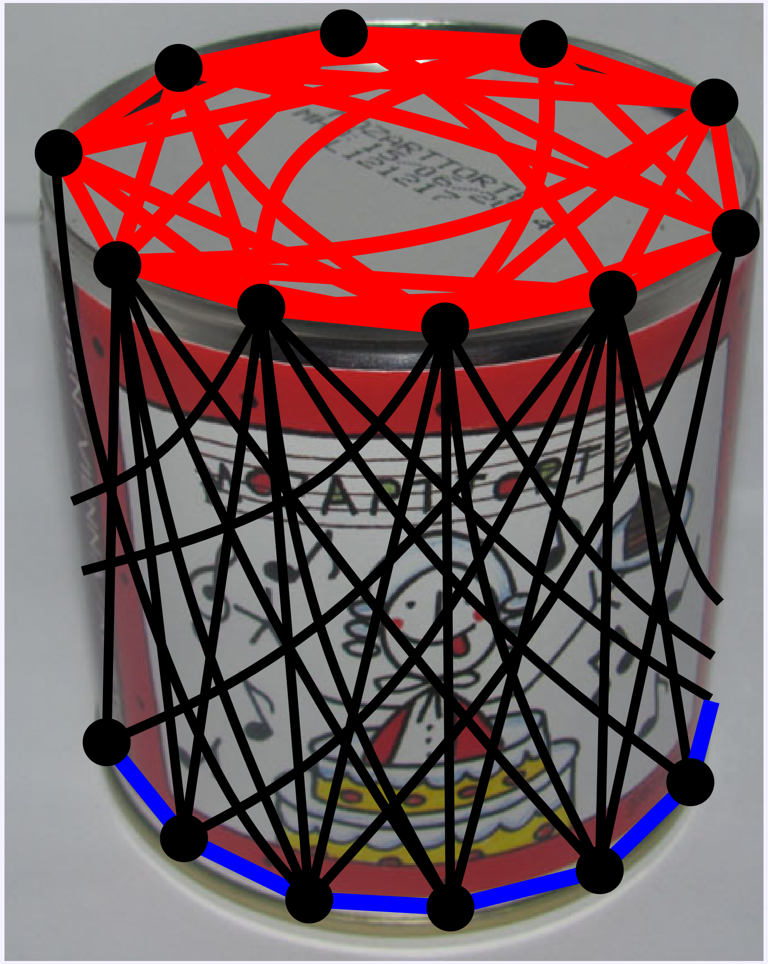


2-page

Cylindrical Drawings



Hill's construction 1963



Shellable

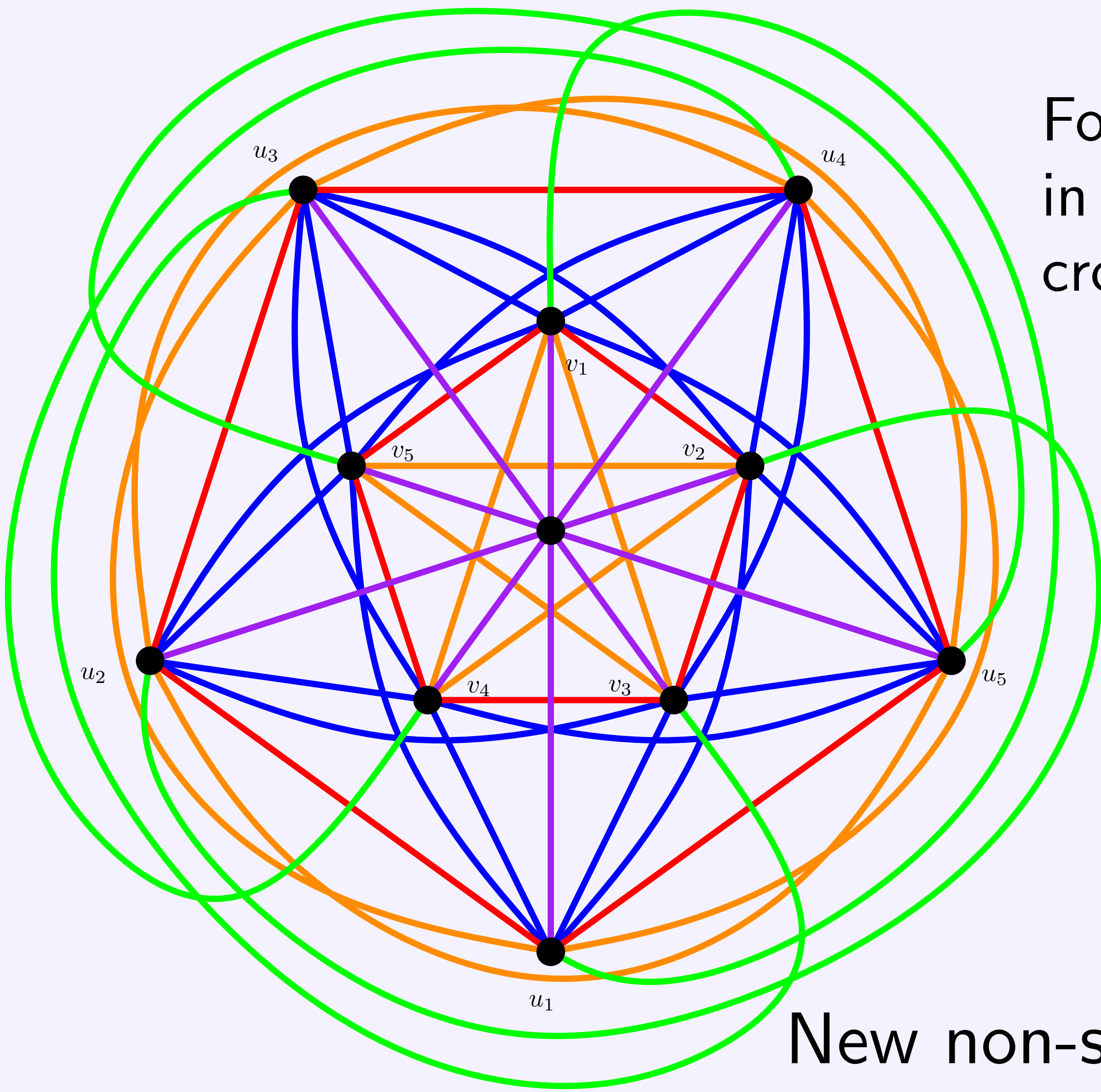
Are all drawings of K_n with $H(n)$ crossings s-shellable?

Or at least all but a constant number of (small) drawings?

No: There exist crossing minimal but non-shellable families of drawings, based on Hill's construction

[Ábrego, A., Fernández-Merchant, Ramos, Vogtenhuber, 2014]

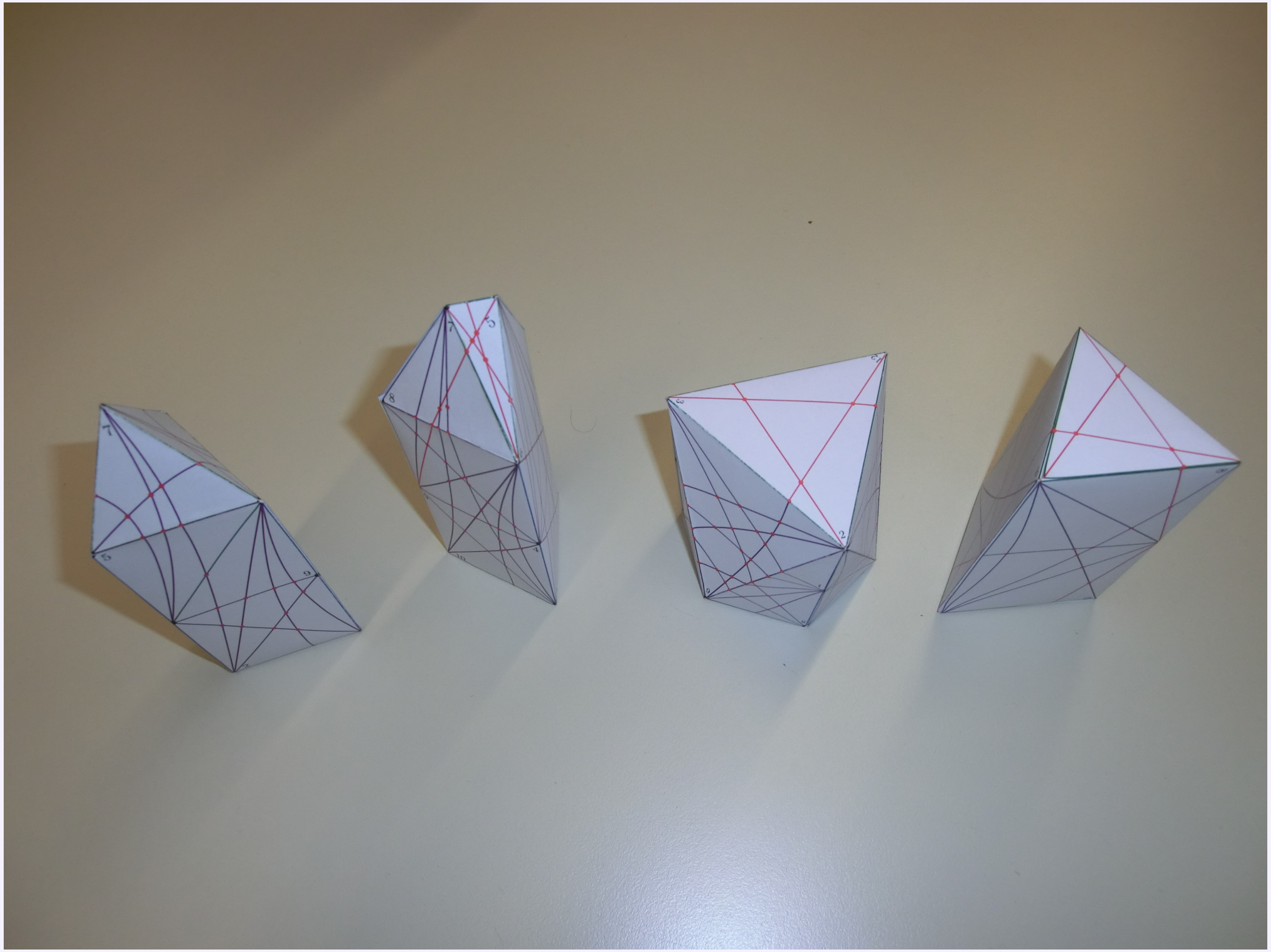
Non-Shellable Family of Point Sets



For $n \geq 11$ all edges in this new family are crossed

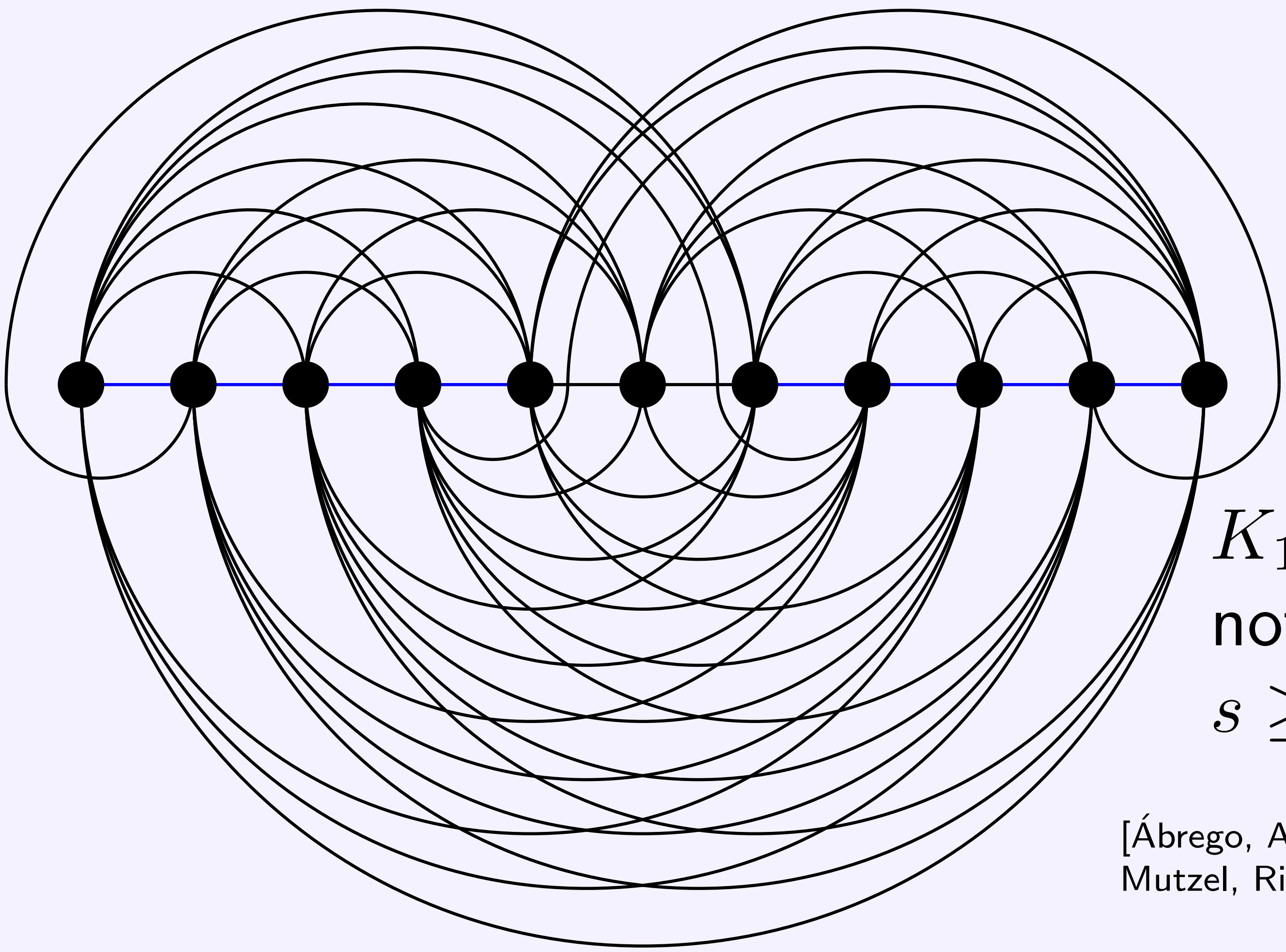
New non-shellable family

Non-Shellable Optimal Symmetric Drawings



Behind Shellability

- Latest concept: bishellable drawings



K_{11}: crossing-optimal, not s-shellable for any $s \geq 5$, but bishellable

[Ábrego, A., Fernández-Merchant, Ramos, Mohar, Mutzel, Richter, Vogtenhuber, 2014]

Crossing Number for Good Drawings

exact crossing numbers for $n \leq 10$ [Guy 1972]

$cr(11) = 100$ and $cr(12) = 150$ [Pan, Richter, 2007]

$217 \leq cr(13) \leq 225$

$cr(13) \geq 219$ [Pan, McQuillan, Richter, 2013]

$cr(13) \geq 223$ [Ábrego, A., Fernández-Merchant, Hackl, Pilz, Ramos, Salazar, Vogtenhuber 2013]

n	3	4	5	6	7	8	9	10	11	12	13
$cr(n)$	0	0	1	3	9	18	36	60	100	150	223/225
# cr-min RS	1	1	1	1	5	3	421	37	403079	2592	1)
shellable	1	1	1	1	5	3	420	29	225769	395	
non-shellable	0	0	0	0	0	0	1	8	177310	2197	
bishellable	1	1	1	1	5	3	420	29	226595	429	
non-bishellable	0	0	0	0	0	0	1	8	176484	2163	

1) There are 9427414 RS with 225 crossings with a sub set of size 12 with 150 crossings.

Decide $cr(13) = 223$ or $cr(13) = 225$

[same group, 6.5 million CPU hours later (2014/15?)]

$cr(13) = 219, 221, 223, 225$?

A drawing with n vertices and few crossings must have a sub drawing of $n-1$ vertices with few crossings:
$cr_{min}(n-1) \leq \lfloor \frac{n-4}{n} cr_{min}(n) \rfloor$

For n odd the crossing number has the same parity for all drawings of K_n.

Extending from 12 to 13:

$cr(13)$	$cr(12)$	status
215	$\leq$ 148	no set for $n = 12$
217	$\leq$ 150	no examle
219	$\leq$ 151	checked 150, 151, no example
221	$\leq$ 153	checked 152, 153, no example
223	$\leq$ 154	checking 154 - still running
225	$\leq$ 155	examples exist

$cr(13) = 219, 221, 223, 225$?

$n = 12$, crossing minimal sets:

cr	150	151	152	153	154	$\leq$ 154
#RS	2592	73014	980495	8137376	46850304	56043781

How to obtain those sets for $n = 12$?

Recurse:

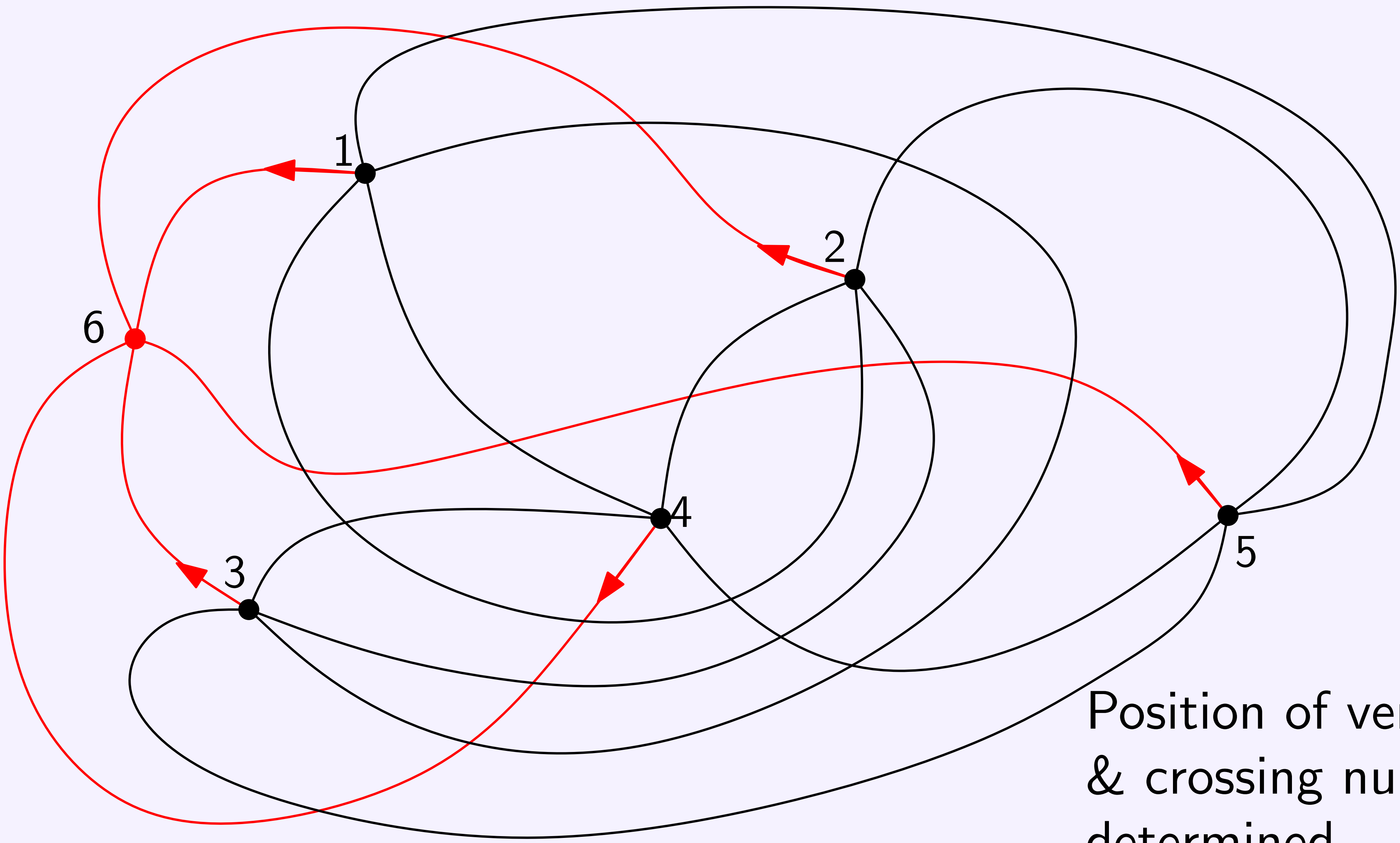
$cr(13)$	$cr(12)$	$cr(11)$	$cr(10)$	$cr(9)$	$cr(8)$
223	$\leq$ 154	$\leq$ 102	$\leq$ 64	$\leq$ 38	$\leq$ 21

$n = 8$, crossing minimal sets:

cr	18	19	20	21	≤ 21
#RS	3	12	50	127	192

(out of 5 370 725 RS)

Extending 8-9-10-11-12-13 / 21-38-64-102-154-223

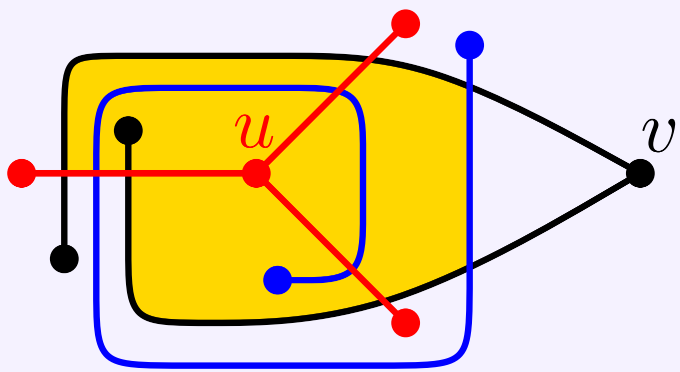


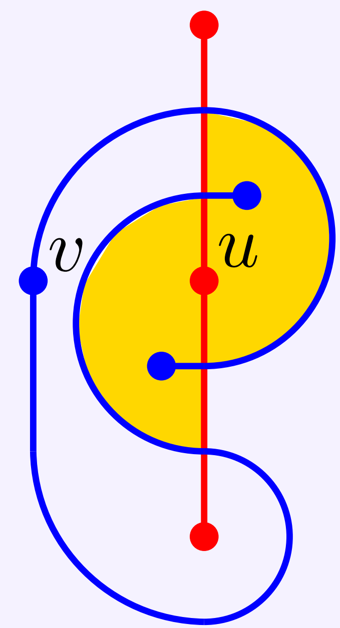
Position of vertex & crossing number determined

Extending Good Drawings

Can any good drawing of a non-complete graph be extended to a good drawing of K_n?

No:

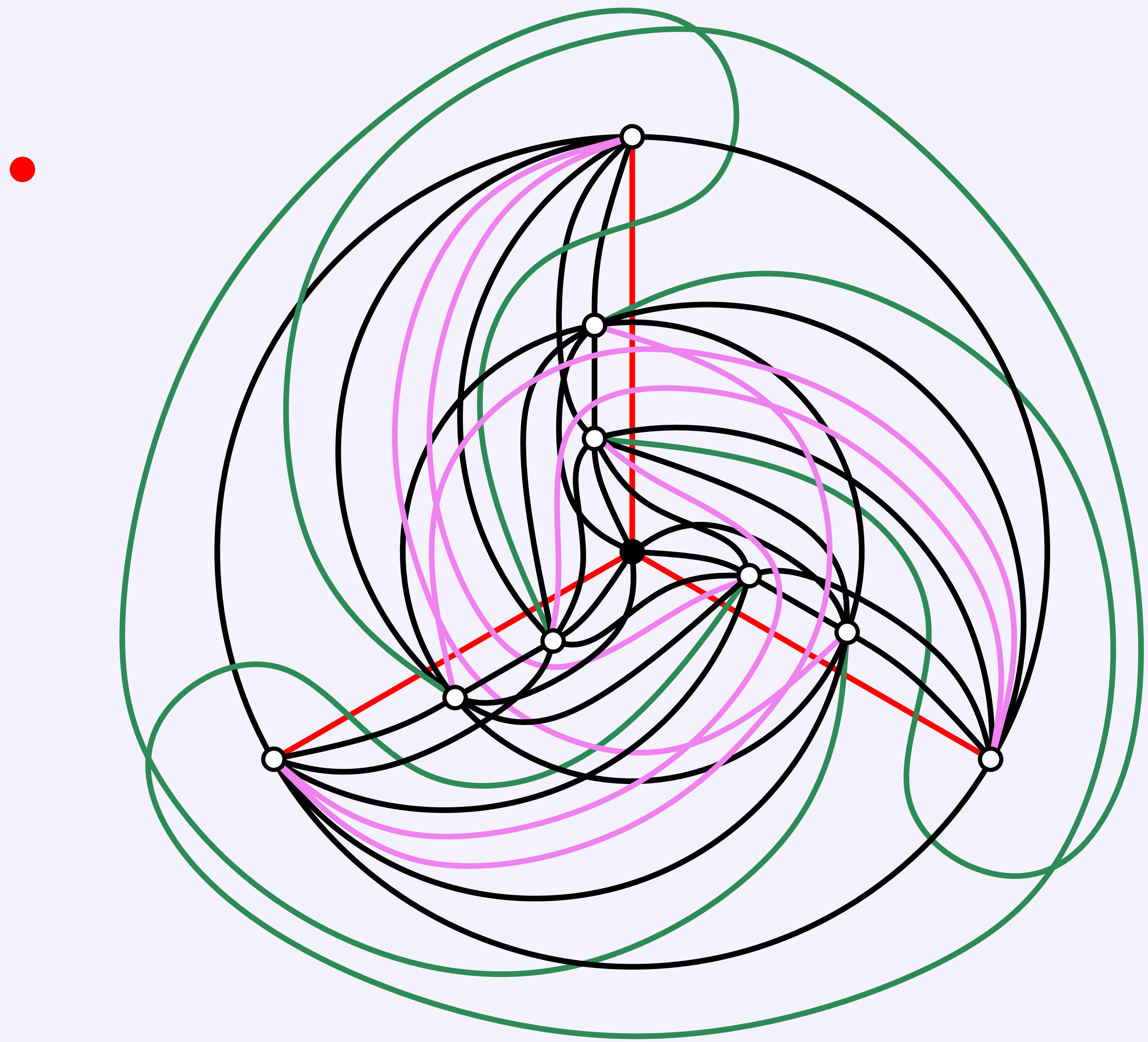




[Kynčl 2013]

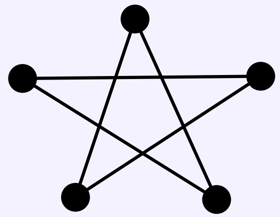
Edge uv can not be part of a good drawing

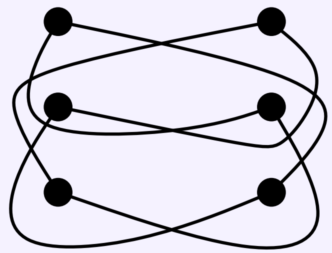
Extending Good Drawings

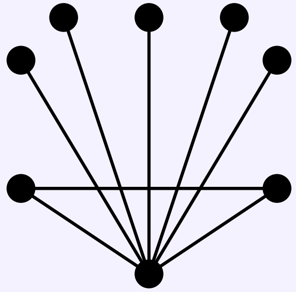


Thrackles

Thrackle (Conway): Good drawing of (non-complete) graph, such that every pair of edges has one point in common (either a common endpoint, or a proper crossing)



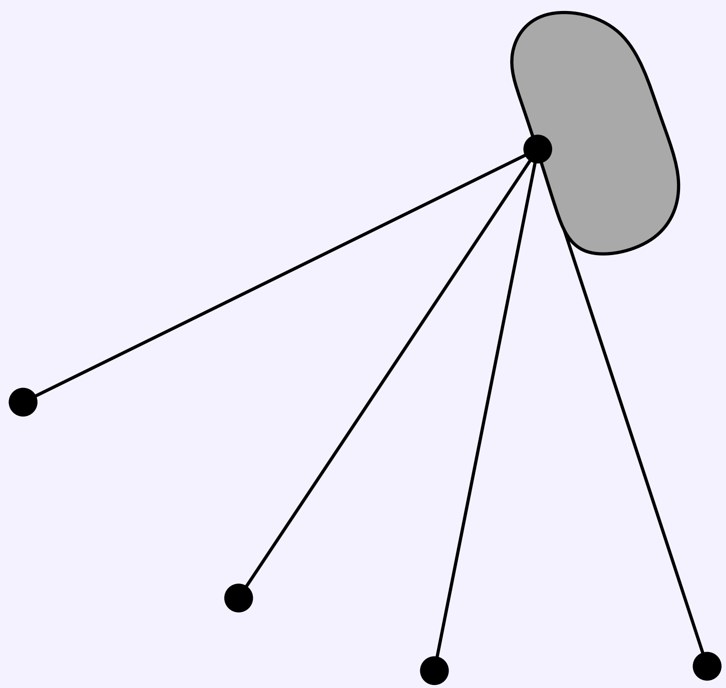


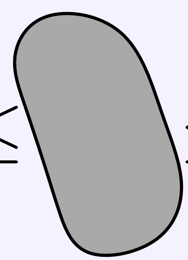


Conway's thrackle conjecture: The number of edges of a thrackle cannot exceed the number of its vertices.

Geometric Thrackles

Conway's thrackle conjecture is true for geometric graphs [Hopf, Pannwitz; Sutherland; Erdős, Perles]



$\Rightarrow$ no. edges $\leq$  $\leq$ no. vertices

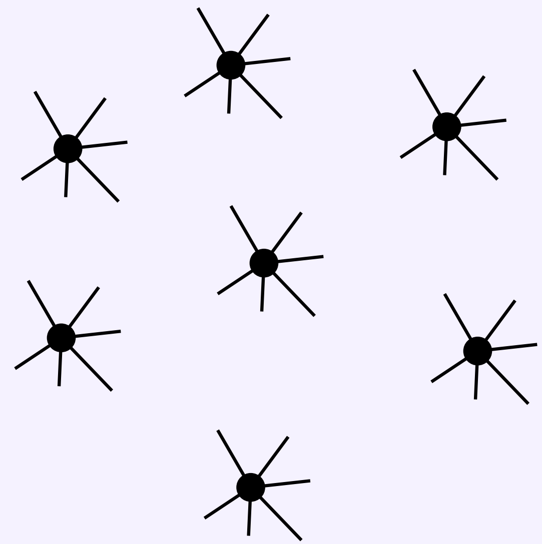
Thrackles

- $t(n) \leq 2n - 3$ [Lovász, Pach, Szegedy 1998]
- $t(n) \leq \frac{3}{2}(n-1)$ [Cairns, Nikolayevsky, 2000]
- $t(n) \leq \frac{167}{117}n < 1.428n$, finite approximation scheme [Fulek, Pach 2010]
- Conjecture true for monotone thrackles [Pach, Sterling 2011
- tangled-thrackles have $O(n)$ edges [tomorrow afternoon, Ruiz-Vargas, Suk, Toth (GD 2014)]

Abstract $(n+1)$-Thrackle

Rotation system:

1: 2 3 4 5 6 7
2: 1 3 4 5 6 7
3: 1 2 4 6 5 7
4: 1 5 7 2 6 3
5: 1 4 2 6 3 7
6: 1 7 2 4 3 5
7: 1 4 6 2 3 5



All 4-tuples realizable $\Rightarrow$ crossing information correct
8 edges 1-3, 1-5, 1-7, 2-4, 2-6, 3-4, 3-7, 5-6 form a thrackle

rotation system non-realizable

(n+1)-Thrackles

Observation: The smallest $(n+1)$-Thrackle contains a spanning path

n	thrackles	tree-thrackles	path-thrackles
2	-	1	1
3	1	1	1
4	1	2	1
5	6	5	2
6	48	41	12
7	994	698	121
8	38 472	22 230	2 399
9	2 580 004	1 166 917	73 092
10	-	-	3 502 013
11	-	-	258 438 398
12	-	-	31 176 142 191

If an $(n+1)$-Thrackle exists, then $n \geq 13$

Open Problems and Future Research

- Prove the Harary-Hill conjecture (ca. 1958):
  $cr(n) \geq Z(n) := \frac{1}{4}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor$
- Do you have a conjecture for the geometric crossing number for K_n?
- Give bounds on the number of bends per edge in a good drawing of K_n for a given rotation system? Can this number easily be determined from the rotation system?
- Does every good drawing of the complete graph K_n contain a plane Hamiltonian cycle?

Grazie

Danke

Merci

Thank you for your attention!

Muchas gracias

Hvala lepa

Dank U wel

どうも ありがとう

Obrigado