Universität Konstanz

Simultaneous Embeddability of Two Partitions

Jan C. Athenstädt[1], Tanja Hartmann[2] & Martin Nöllenburg[2]

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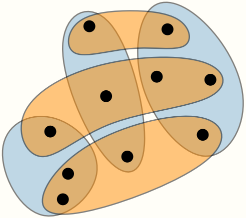


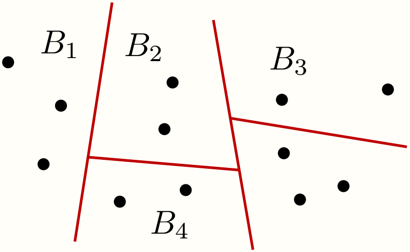
GD 2014 - September 24th, 2014

Introduction

Partitions

definition: partition of a finite universe U

- $\mathcal{P} = \{B_1, \ldots, B_n\}$ collection of subsets ("blocks") of U
- every $u \in U$ contained in exactly one $B \in \mathcal{P}$



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- induced by parameter of a dataset
  - multiple independent parameters possible
- result of a clustering algorithm
  - different algorithms return different results

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How can we compare two partitions?

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Related Work

- numeric measures of similarity for two partitions [Wagner & Wagner 2007]
  - does not show where the differences or similarities are

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our contribution

- classification of *simultaneous embeddings* of two partitions

Introduction

Embeddings

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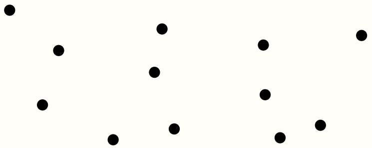
embedding Γ *of* $\mathcal{S} \subseteq 2^U$ *maps*

- $u \in U \to \Gamma(u) \in \mathbb{R}^2$



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- $S \in \mathcal{S} \to \Gamma(S) \subset \mathbb{R}^2$ *such that*
  - $\Gamma(S)$ *is simple, bounded, and closed region*
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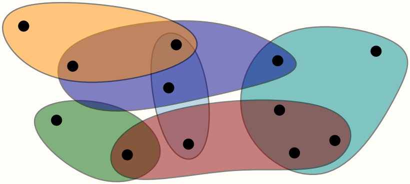
  - *boundaries intersect in true crossing points*



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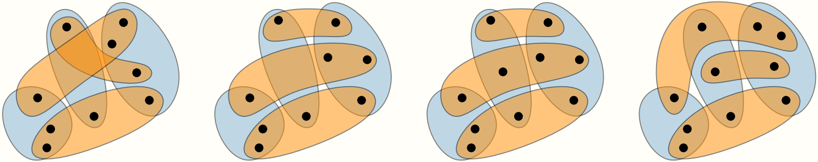
two partitions $\mathcal{P}_1$ and $\mathcal{P}_2$

- (simultaneous) embedding := embedding of $\mathcal{P}_1 \cup \mathcal{P}_2$

Embeddability Classes

Overview



examples of simultaneous embeddings of two partitions

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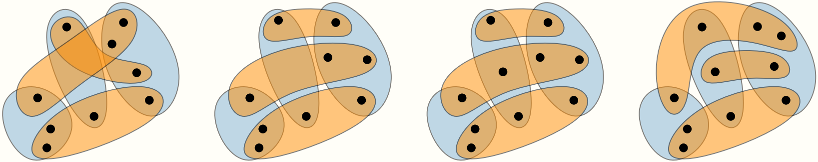


examples of simultaneous embeddings of two partitions

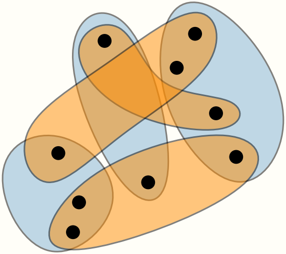
How to classify a "good" embedding?

Embeddability Classes

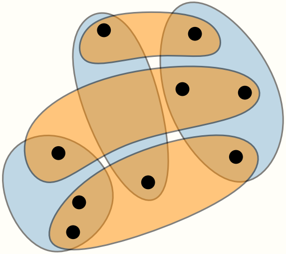
Weak Embeddability

definition: weak embedding

no two block regions of the same partition intersect



non-weak embedding



weak embedding

Embeddability Classes

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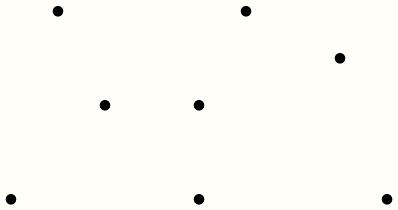

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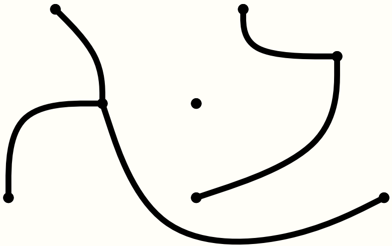

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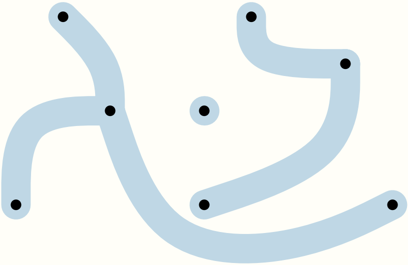

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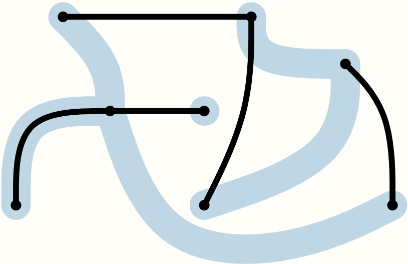

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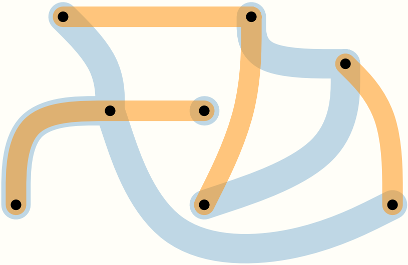
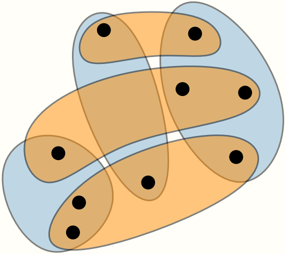


Embeddability Classes

Strong Embeddability

definition: strong embedding

weak embedding + each connected component of the intersection of two block regions contains at least one element



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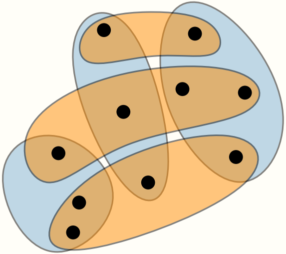

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  - only because $(U, \mathcal{P}_1 \cup \mathcal{P}_2)$ is 2-regular hypergraph
  - equivalent to existence of *planar support* ($\rightarrow$ later) [Kaufmann et al. 2009]

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$$\{\mathcal{P}_1, \mathcal{P}_2\} \text{ strongly embeddable}$$

$$\Leftrightarrow$$

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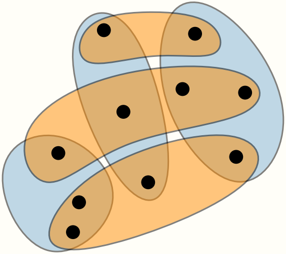
Embeddability Classes

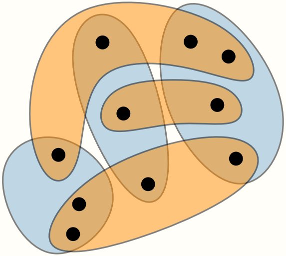
Full Embeddability

definition: full embedding

strong embedding + the boundaries of two block-regions have at most two points of intersection



strong embedding



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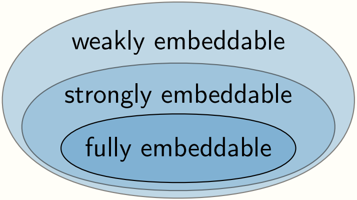
theorem

$$\{\mathcal{P}_1, \mathcal{P}_2\} \text{ fully embeddable}$$
$$\Leftrightarrow$$
$$(U, \mathcal{P}_1 \cup \mathcal{P}_2) \text{ has planar bipartite map}$$

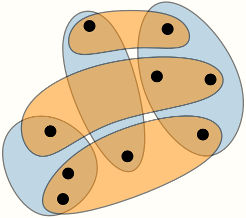
Embeddability Classes

Hierarchy of Embeddability

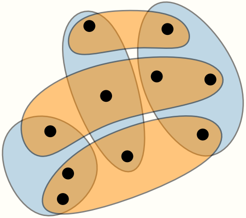


theorem

The hierarchy is strict.



weak embedding



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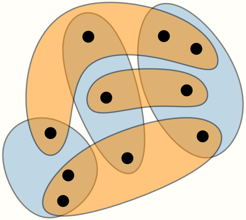

full embedding

Complexity results

NP-completeness of Strong Embeddability

definition: support of a hypergraph [Kaufmann et al. 2009]

- $H = (U, \mathcal{S})$ *is hypergraph with* $\mathcal{S} \subseteq 2^U$
- *support: graph* $G = (U, E)$ *on* U
- *induced subgraph* $G[S]$ *for every hyperedge* $S \in \mathcal{S}$ *connected*

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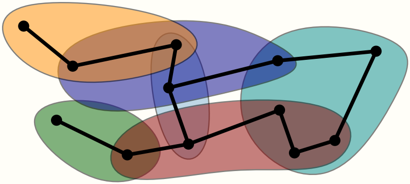

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theorem (reminder)

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$\Rightarrow$ implies NP-completeness of deciding vertex planarity for 2-regular hypergraphs

Complexity results

NP-completeness of Strong Embeddability

theorem

Deciding strong embeddability is NP-complete.

sketch of proof

show that finding a planar support is NP-complete

- membership in NP
  - guess support graph
  - check planarity and support-property in polynomial time
- NP-hardness
  - reduction from PLANAR-MONOTONE-3-SAT
  - inspired by more general proof from [Buchin et al. 2010]

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definition: PLANAR-MONOTONE-3-SAT

3-SAT formula with planar monotone rectilinear representation (MRR)

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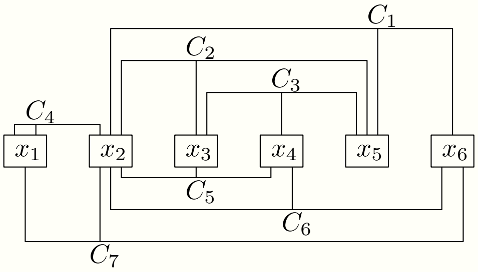

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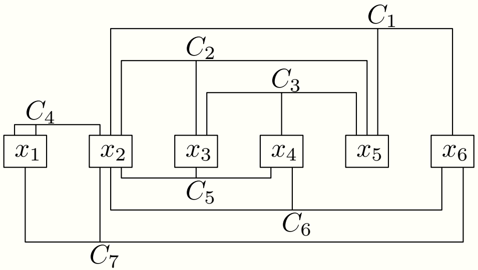

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- NP-complete problem [de Berg & Khosravi 2010]

Complexity results

NP-completeness of Strong Embeddability

given an MRR Φ

- fix clusters on a grid to follow structure of Φ
- inspired by the proof in [Chaplick et al. 2012]

Complexity results

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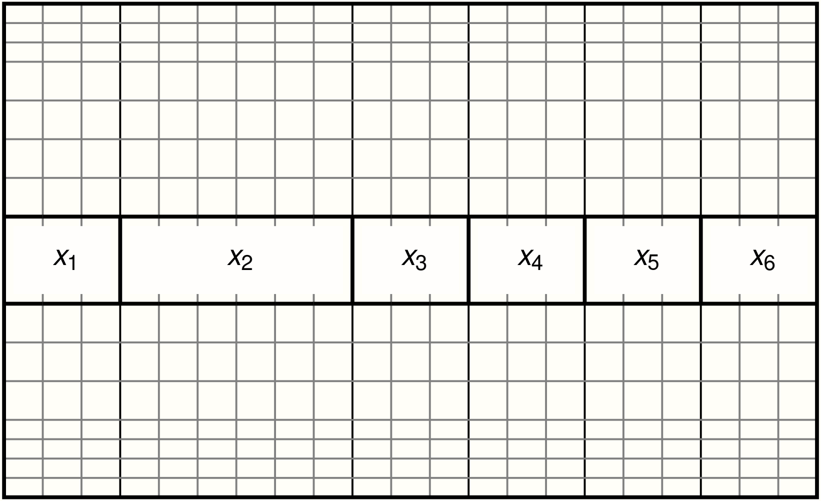

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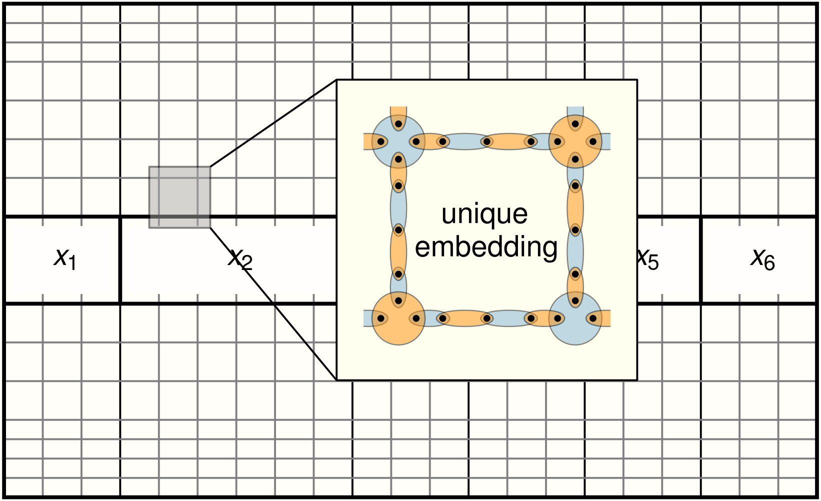

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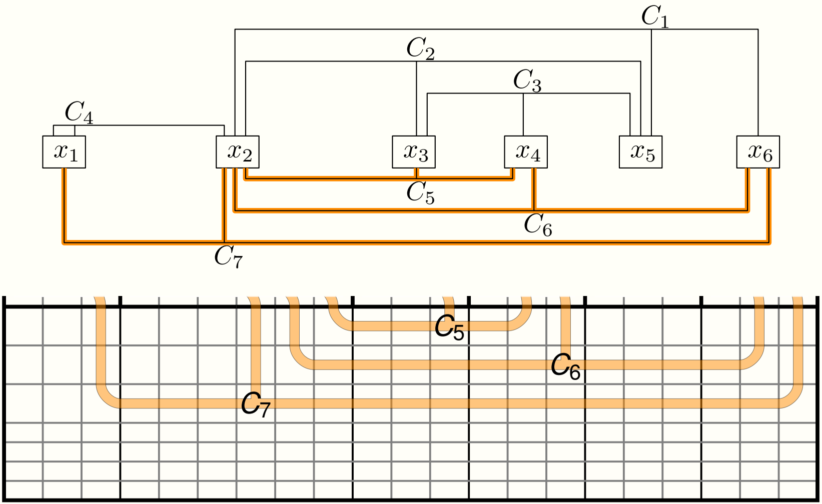

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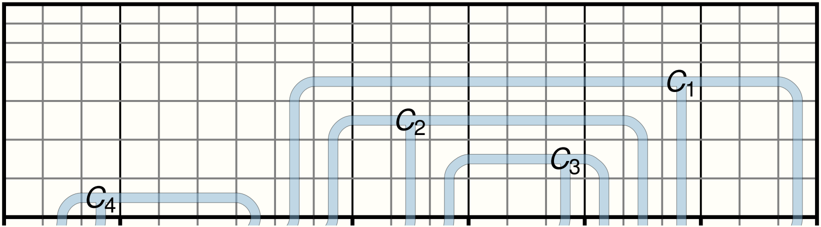

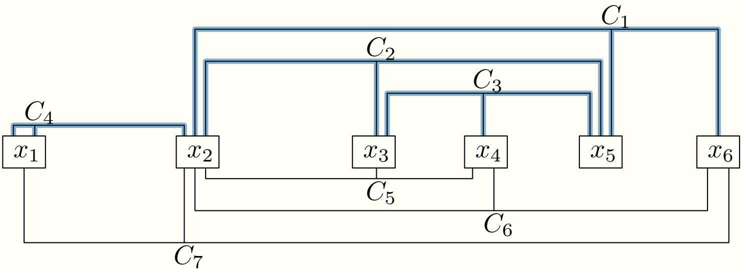

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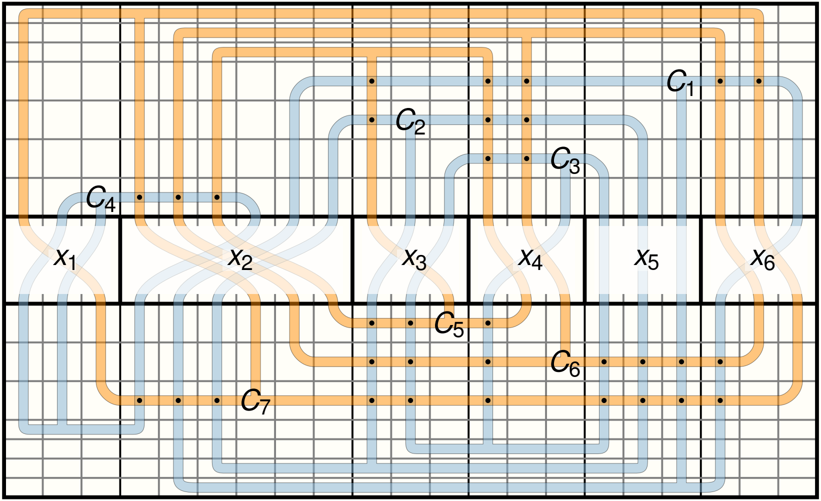

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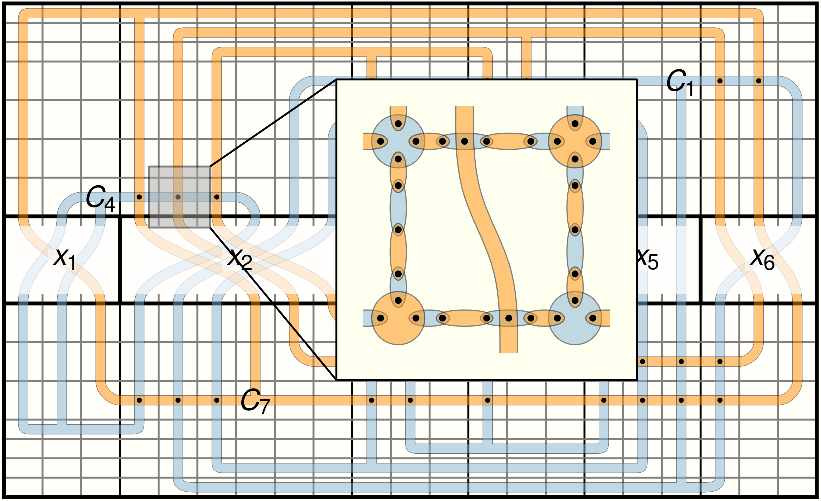

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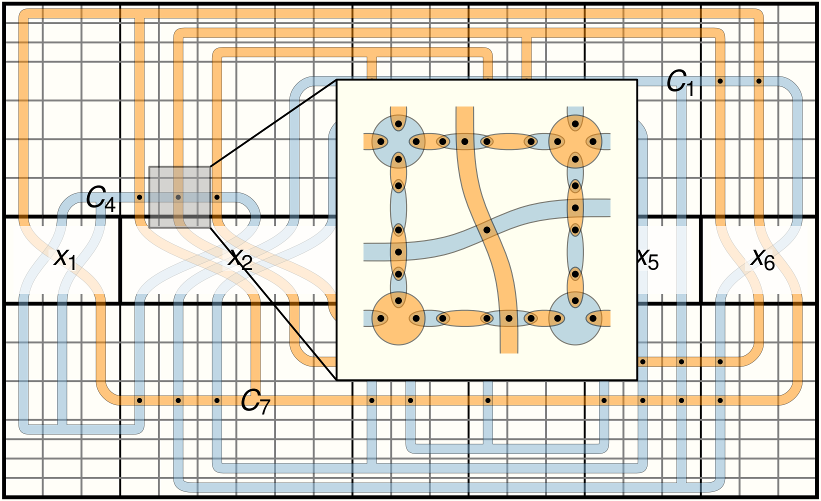

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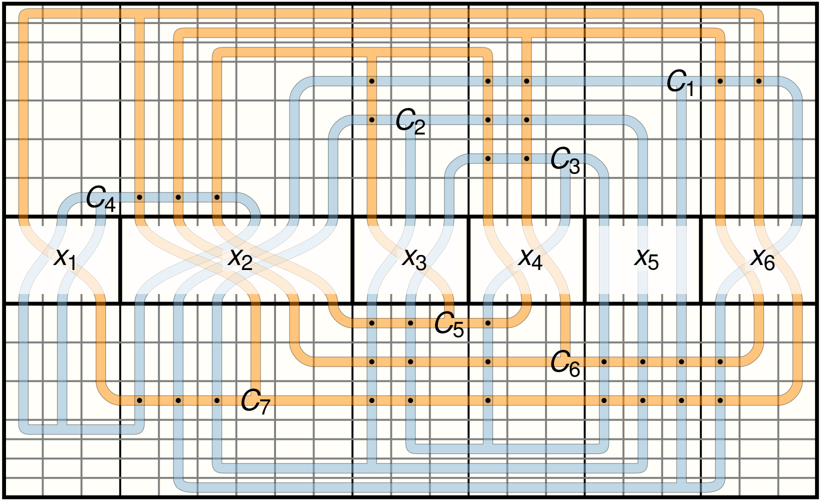

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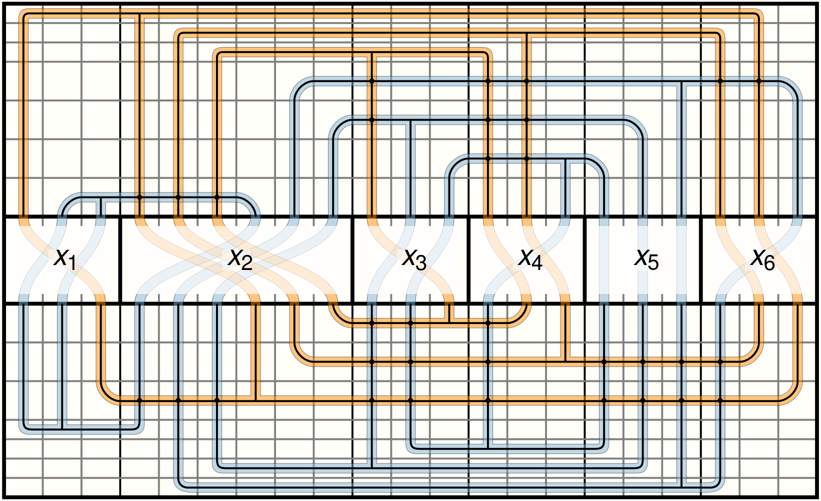

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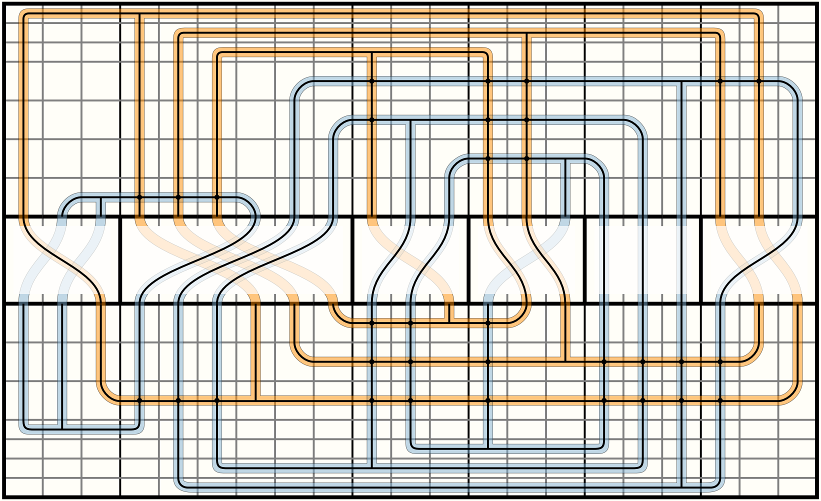

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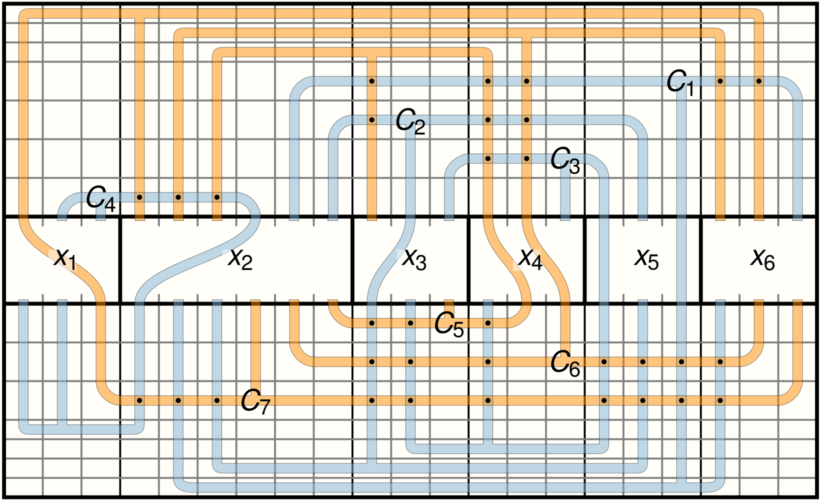
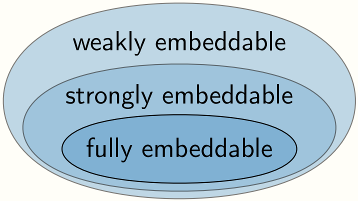


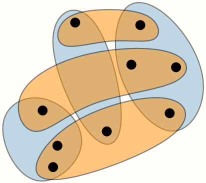
Thank you!

Results and Extensions

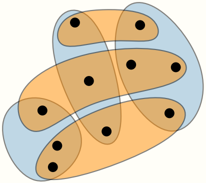


future work

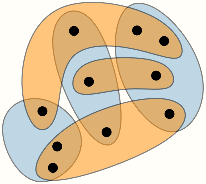
- more than two partitions
- algorithms for visually appealing embeddings
- respect an underlying graph structure



weak embedding
$\Rightarrow$ exists always



strong embedding
$\Rightarrow$ NP-complete



full embedding
$\Rightarrow$ check in lin. time