Flat Foldings of Plane Graphs with Prescribed Angles and Edge Lengths

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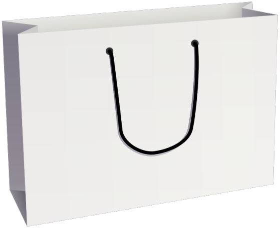
Graph Drawing 2014

Why is it useful to flatten things?

Many situations in which items can be stored or transported more easily when folded into a more compact configuration, or can be manufactured by folding from flat materials

Automotive airbags

Flat-packed furniture



Shopping bags

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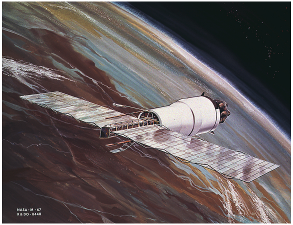
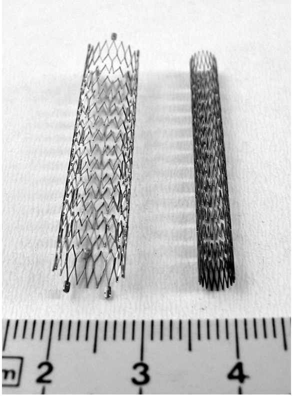


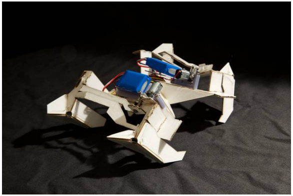
Space missions

PD artist's conception of Pegasus meteoroid detection satellite



Surgical devices

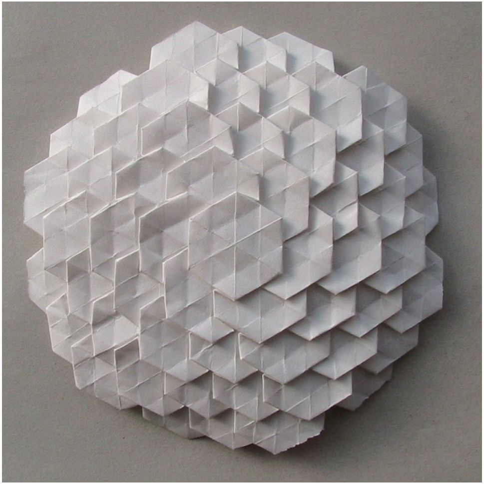
CC-BY-SA image Stent4 fcm.jpg by Frank C. Müller, Wikimedia commons



Self-folding robots

MIT News, August 2014
Photo: Harvard's Wyss Institute

Flattening things that are already flat

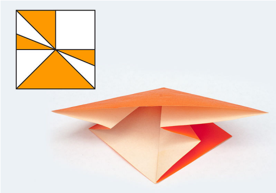


Flat origami: an initially-planar piece of paper is folded into a different state that still lies flat in a plane

Mathematics of flat origami

It's NP-complete to test whether a folding pattern can fold flat

[Bern and Hayes 1996]



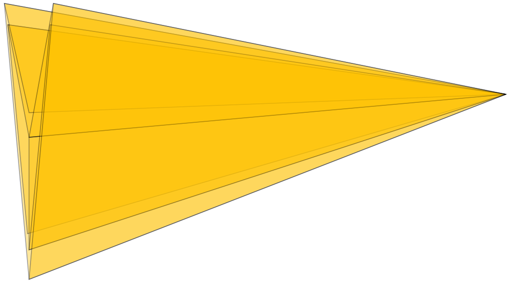
But if there's only one vertex where all fold lines meet, then...

- Flat foldability is polynomial [Bern and Hayes 1996]
- Maekawa's theorem: $|\#\text{ mountain folds} - \#\text{ valley folds}| = 2$
- Kawasaki's theorem: two alternating sums of angles are equal
- Any folded state can be reached by a continuous motion [Connelly et al. 2003; Streinu and Whiteley 2004]

But what if it's not already flat?

The multi-vertex case is still NP-hard

If we don't know which folds are mountain folds and which are valley folds, then even with one vertex the problem is strongly NP-hard [Abel et al. 2013]

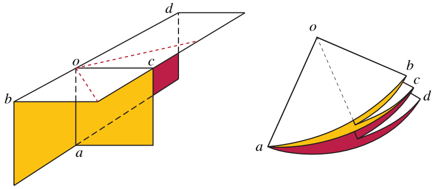


"Locked" states unreachable by continuous motions may exist [Ballinger et al. 2009; Biedl et al. 2002; Connelly et al. 2002]

Our results

Given a two-dimensional complex in which

- All folds must be along edges of the complex
- All folded edges share a common vertex
- Pairs of adjacent faces on the same edge are marked with their target angle: 0, π, or 2π

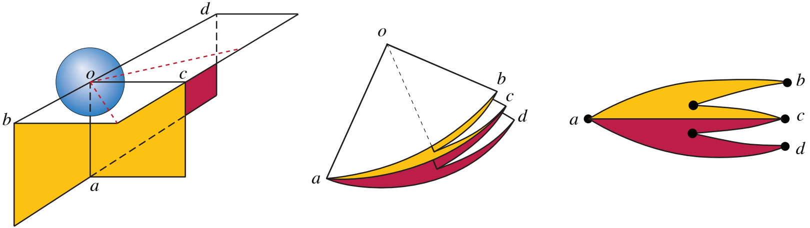


In linear time we can test whether it has a flat-folded state

In polynomial time we can count all flat-folded states

Dimension reduction

Intersect the complex with a small ball near the vertex

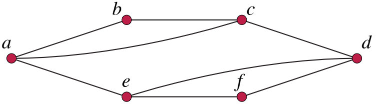


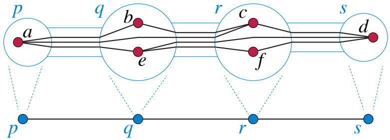
Becomes a one-dimensional graph drawing problem:
finding *flat embeddings* of *plane graphs*

Self-touching configurations

How to describe a flat embedding?

What does it mean for such an embedding to be non-crossing?





Self-touching configuration [Connelly et al. 2003; Ribó Mor 2006]:

map from a given plane graph to a path

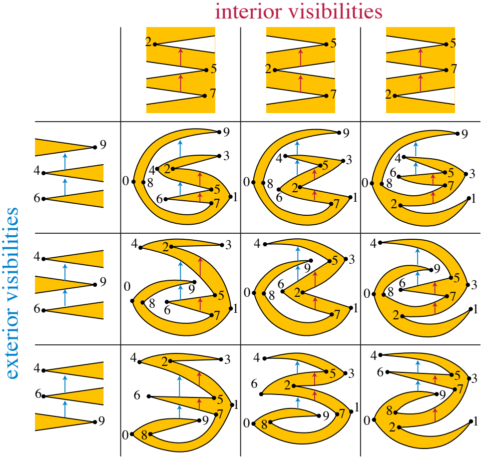
together with magnified views of the path vertices and edges

Face independence

Main technical lemma:

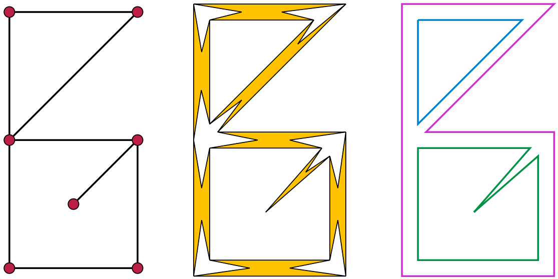
G can be flattened if and only if each face of G can be flattened

The number of flat foldings of G is the product of the numbers of flat-folded states of each face



Euler tours

In the given planar embedding, not all faces may be simple cycles...

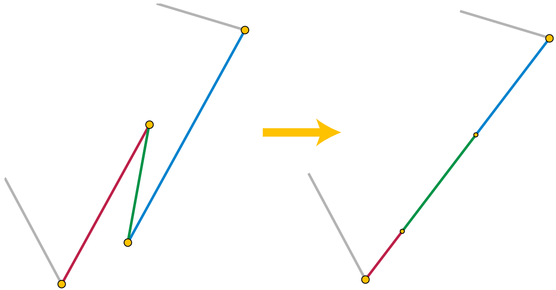


...but we can convert them to cycles by using an Euler tour, without changing foldability

Greedy crimping

To test whether a single face cycle has a flat-folded state, repeatedly:

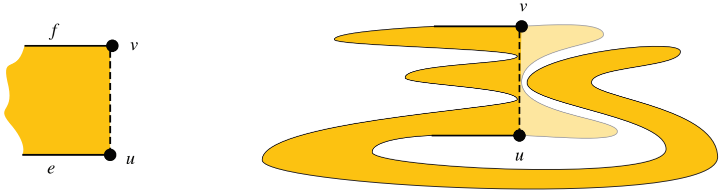
- Find an edge of locally-minimum length
  with opposite-type folds at its endpoints
- Glue it to its neighbors, reducing the complexity of the cycle



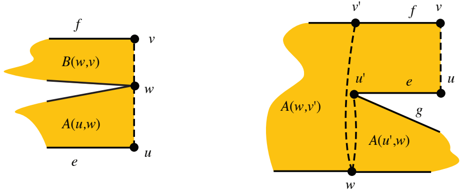
[Arkin et al. 2004; Bern and Hayes 1996; Demaine and O'Rourke 2007]

Dynamic programming

Can count folded states of a cycle by finding pairs of vertices (u, v) that can be visible to each other with same coordinate, forming smaller subproblems in which they are glued together



(a) (b)



(a) (b)

Conclusions

Can test flat-foldability of one-vertex complexes by reducing dimension to planar graph problem, finding Euler tours of faces, applying greedy crimping to each face

Same method + dynamic programming works for counting flat-folded states

Similar counting algorithms likely apply to many graph drawing problems with analogous face-independence properties (upward planar embeddings, level planar embeddings, ...)

Version where angles between adjacent faces are unspecified but must be in $\{0, 2\pi\}$ (no flat angles allowed) is **still open**

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