Disjoint edges in topological graphs and the tangled-thrackle conjecture

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Graph Drawing 2014

Thrackles

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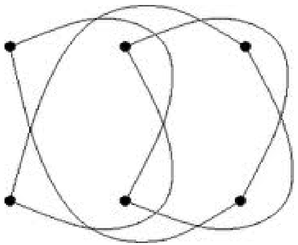
- Every pair of edges meets exactly once: at a vertex or at a crossing point.

State of affairs

- Conway's conjecture: If a thrackle has n vertices then it has at most n edges.
- If true, it would be tight: every cycle with more than 4 vertices can be drawn as a thrackle.



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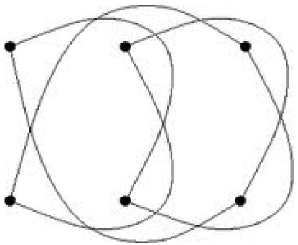

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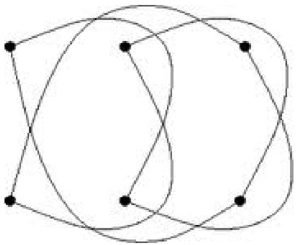

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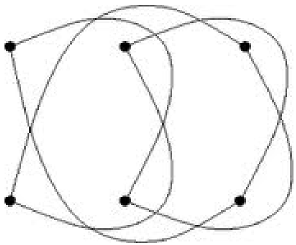
- (2011)Fulek, Pach: 1.428n



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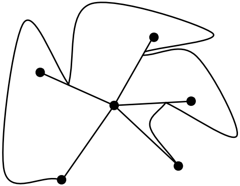

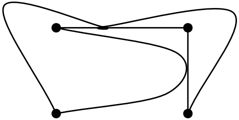

Theorem

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(Pach, Tóth, Radoičić, 2011) A tangle with n vertices has at most n edges.

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- Every pair of edges meets exactly once: at a a vertex, at a crossing or at a touching.
- Touching and crossing points are all distinct.
- What is the maximum number of edges tangled-thrackle with n vertices can have? (Pach, Radoičić, and Tóth)

Tangled-thrackles

- $O(n \log^{12} n)$ (Pach, Radoičić, and Tóth, 2012).

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- We proved their conjecture.

A first observation: Bounding number of touchings

- No 200 edges touch another set of 200 edges.

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- No 200 edges touch another set of 200 edges.
- I.e. the touching graph has no $K_{200,200}$. By Kövári, Sós, Turán number of touchings is at most

$$c|E|^{2-1/200} \leq c(n\log^{12} n)^{2-1/200} \leq cn^{2-1/1000}$$

Odd-crossing number

Definition

The odd-cr(G) is the least number of pairs of edges that cross an odd number of times among all drawings of G.

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The bisection width $b(G)$ is the least number of edges from V_1 to V_2 among all partitions V_1, V_2 of V with $V_i \geq n/3$.

Odd crossing number

Theorem

(Pach, Tóth) There is an absolute constant c_2 such that if G is a graph with n vertices of vertex degrees $d_1, \ldots, d_n$, then

$$b(G) \leq c_2 \log n \sqrt{\textit{odd-cr}(G) + \sum_{i=1}^{n} d_i^2}.$$

Redrawing

- How do we use this theorem?

Redrawing

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- We assume G is bipartite. Whichever edges are touching we changed them slightly so that they become disjoint.

Redrawing

- We show that if G is drawn as tangled thrackle then its odd-crossing number is small.

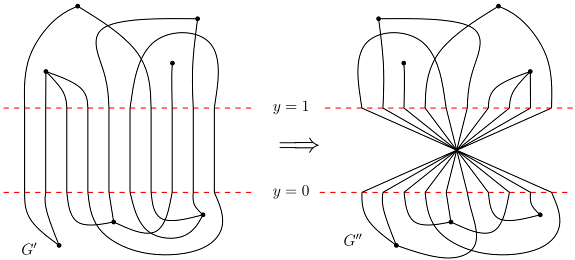

Figure : Redrawing procedure

Redrawing

- After redrawing a pair of edges crosses an odd number of times if and only if they were originally touching.

Bounding number of touchings

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- $b(G) \leq n^{1-1/2000}$

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- Small bisection width: bound odd crossing number.
- Redrawing and Kövári, Sós, Turán.
- Decompose the graph into two parts using small bisection width and apply induction.
- Show that a tangled thrackle has at most $c(n - n^{1-1/4000})$ edges.

Some open questions

- What is the smallest t such that there is no $K_{t,t}$ on the touching graph?

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- What is the smallest t such that there is no $K_{t,t}$ on the touching graph?
- Thrackle conjecture is still open.

Thank you.