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Overlapping Cluster Planarity.

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Abstract

This paper investigates a new direction in the area of cluster planarity by addressing the following question: Let G be a graph along with a hierarchy of vertex clusters, where clusters can partially intersect. Does G admit a drawing where each cluster is inside a simple closed region, no two edges intersect, and no edge intersects a region twice? We investigate the interplay between this problem and the classical cluster planarity testing problem where clusters are not allowed to partially intersect. Characterizations, models, and algorithms are discussed.

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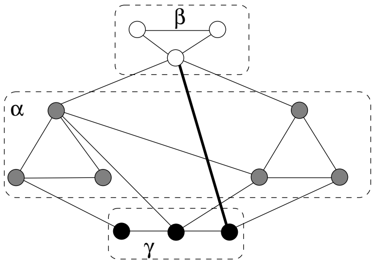
1 Introduction

Graphs and their visualizations are essential in data exploration and understanding, particularly for those applications that need to manage, process, and analyze huge quantities of relational data. However, when the graph to be displayed consists of hundreds or thousands of vertices, a complete visualization of the data is typically not effective for the user, and therefore alternative visualization paradigms have been investigated in the literature. A well studied approach to handle and visualize large graphs is to organize the vertices into a *hierarchy of clusters*. This makes it possible to explore complex relational data at different levels of detail, by collapsing or expanding clusters. This approach has been applied to various application domains, including Internet and Web computing, social network analysis, reverse engineering, knowledge engineering, and computational biology (see, e.g., [8, 15, 16, 20]).

A *clustered graph* (or simply *c-graph*) consists of a pair $C = (G, T)$, where G is an undirected graph and T is a rooted tree that describes a hierarchy of vertex clusters; each cluster is a subset of the vertices of G and any two clusters are either disjoint or one is completely included in the other. In a visualization of a c-graph the subgraph induced by each cluster α is drawn inside a simple closed region which keeps any other vertex that does not belong to α out of it. Also, the inclusions among cluster regions must reflect the inclusion relations among the corresponding clusters. An important requirement for the readability of the drawing is that it has as few crossings as possible: It is required to minimize both crossings between edges and crossings between a cluster region and edges that are not incident to vertices inside the region. A crossing-free drawing of a c-graph is called a *c-planar drawing* and a c-graph that admits such a drawing is said to be *c-planar*. For example, Figure 1 shows two different drawings of the same clustered graph, where the regions of the clusters are drawn as rectangles. The drawing in Figure 1(a) is not c-planar, because the bold edge crosses both other edges and the region of cluster α. Conversely, the drawing in Figure 1(b) is c-planar.

The problem of testing whether a c-graph is c-planar was first introduced in a paper by Feng, Cohen, and Eades [9], that inspired and motivated a sequence of papers on this topic. Feng et al. [9] describe a quadratic-time c-planarity testing algorithm for clustered graphs where each cluster induces a connected subgraph. Linear-time testing algorithm for the same class of clustered graphs are described in [2, 4, 5]. Feng et al. leave as open the problem of testing a c-graph for c-planarity when clusters can induce non connected subgraphs; although the time complexity of this problem is still unknown, several special cases for which polynomial-time testing algorithms exist have been described in the literature [3, 10, 11]. The relationship between planarity and c-planarity has also been studied in [1] and a planarization algorithm for c-graphs that are not c-planar is described in [6].

Motivated by the several applications where relational data are clustered and the clusters can partially intersect (see, e.g., [13, 14, 20]), recent papers by



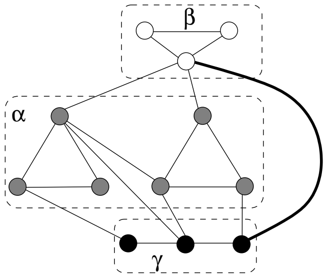


Figure 1: (a) A non c-planar drawing of a clustered graph. (b) A c-planar drawing of the same clustered graph.Vertices with the same color belong to the same cluster.

Omote and Sugiyama [17, 18] present effective generalizations of force-directed techniques to compute drawings of clustered graphs where distinct clusters can share subsets of their vertices. The methods proposed in [17, 18] can however give rise to drawings that do not respect some of the classical rules of cluster planarity; for example, two edges may intersect each other or an edge may intersect the region of a cluster that contains none of its end-vertices.

Inspired by the above mentioned cluster planarity literature and by the work of Omote and Sugiyama [17, 18], this paper opens a new research direction in the field of cluster planarity testing by addressing the following problem: Let G be a graph along with a hierarchy of vertex clusters, where clusters can *overlap*, i.e., they can share a proper subset of their vertices; does G admit a drawing where each cluster is inside a simple closed region, no two edges intersect, and no edge intersects a region twice? Figure 3(a) depicts a drawing of an overlapping clustered graph that satisfies the desired conditions. At a first glance, one might argue that the question of this paper can be answered by simply regarding each overlap between any two clusters as an individual cluster and by applying known results of cluster planarity. However, as it will be shown throughout the paper, this approach does not work in general even for the apparently simple case of a graph consisting of exactly two overlapping clusters. Indeed, the main focus of this paper is on the study of the relationship between cluster planarity with overlaps and cluster planarity without overlaps.

An overview of the main results in this paper is as follows.

- We define the concept of *overlapping clustered graphs* (also called *oc-graphs*) and of *overlapping cluster planarity.* An oc-graph that is cluster planar is called *oc-planar.* A characterization of oc-planar graphs is given; this characterization extends the one of Feng et al. [9] to the case of overlapping clusters.

- We provide examples of oc-planar oc-graphs such that the c-graph obtained by considering the overlaps as individual clusters is not c-planar; and, vice-versa, we give examples where the oc-graph is not oc-planar while the corresponding c-graph is c-planar.
- Based on the above characterization and negative results, we describe models that make it possible to translate the planarity testing problem for meaningful classes of oc-graphs as the planarity testing problem of associated c-graphs.
- Planarity testing and embedding algorithms for oc-graphs are devised by combining our models with known results about c-planarity.

We recall that a generalization of clustered graphs that includes the family of oc-graphs, known as *compound graphs*, has been introduced in the graph drawing literature several years ago by Sugiyama and Misue [21]. In a compound graph clusters may overlap and adjacency relations among clusters may be defined (in other words, there may be edges connecting pairs of clusters and not just pairs of vertices). It has to be remarked however that, to the best of our knowledge, all drawing algorithms provided in the literature to visualize compound graphs, including the techniques in [21], work under the restrictive assumption that no two clusters overlap (see, e.g., [15, 16, 20]). Also compound graphs have never been studied from a planarity testing perspective.

The remainder of this paper is organized as follows. The definition of oc-graphs is given in Subsection 2.1 and a characterization of oc-planarity is presented in Subsection 2.2; models and algorithms can be found in Section 3; final remarks and open problems are in Section 4.

2 Overlapping Clustered Graphs and Planarity

We first define oc-graphs and oc-planarity and then present a characterization result. We assume familiarity with basic concepts of graph theory [12] and geometric computing [19]; we recall here only those definitions that pertain cluster planarity.

2.1 Definitions

A graph G is *connected* if it consists of one vertex only, or if for any pair of its vertices u and v there exists a path connecting u to v in G. A connected graph G is *k-connected* ($k > 1$) if it remains connected after the removal of any subset of $k - 1$ vertices.

A *clustered graph* $C = (G, T)$, also called *c-graph*, consists of an undirected graph G and of a rooted tree T, called *inclusion tree of* C, which describes the inclusion relationships among the vertex clusters. Namely:

- The leaves of T are the vertices of G;

- Let μ be an internal node of T. Node μ has at least two children and it represents a *cluster* of vertices of G, denoted as $V(\mu)$. Cluster $V(\mu)$ consists of all leaves of the subtree rooted at μ. If ν is an ancestor of μ in T, then $V(\mu) \subset V(\nu)$.

Figure 2 shows an example of a c-graph. In the remainder of the paper $G(\mu)$ denotes the subgraph of G induced by $V(\mu)$. A c-graph C is said to be *c-connected* if $G(\mu)$ is a connected graph for each μ of T. The c-graph of Figure 2(a) is not c-connected, because both clusters α_2 and β_2 induce a disconnected subgraph of G.

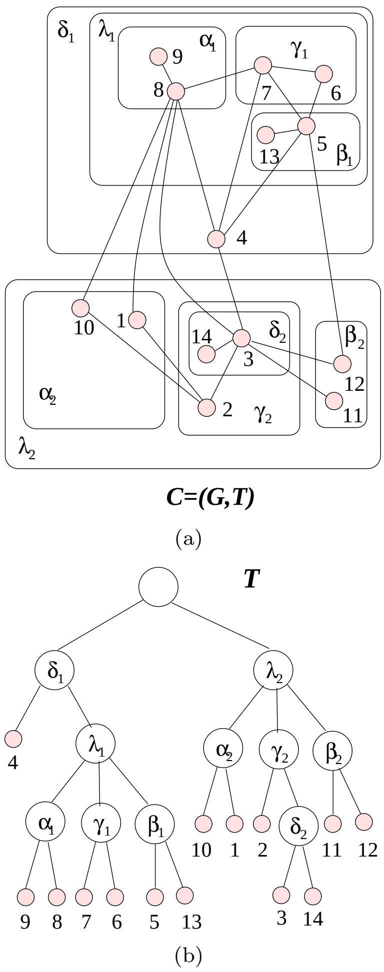


Figure 2: (a) A c-graph $C = (G, T)$; the clusters are represented as rectangles. (b) The inclusion tree T of C.

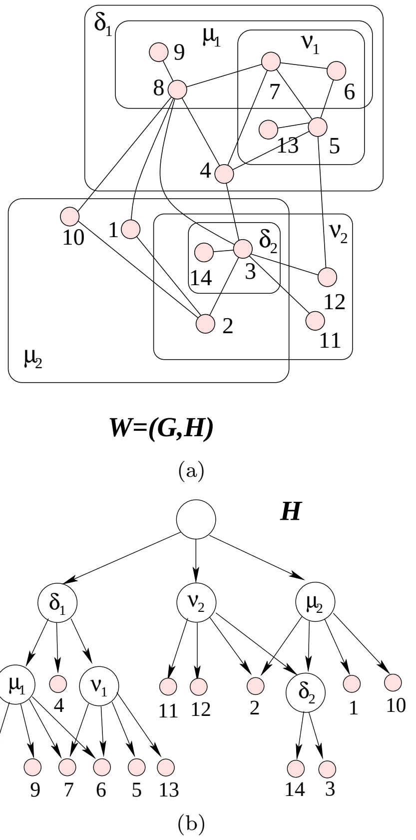


Figure 3: (a) An oc-graph W; the clusters are depicted as rectangles. Clusters μ_1, ν_1 and clusters μ_2, ν_2 are overlapping clusters. (b) The inclusion digraph of W.

An *overlapping clustered graph* $W = (G, H)$ consists of an undirected graph G and an acyclic digraph H with a single source such that:

- The sinks of H are the vertices of G;
- Let μ be a non-sink node of H. Node μ has at least two outgoing arcs and it represents a *cluster* of vertices of G, denoted as $V(\mu)$. Cluster $V(\mu)$ consists of all sinks of H reachable from μ with a directed path;
- There are no transitive arcs in H.

In the following we call W an *oc-graph* and H the *inclusion digraph* of W. Figure 3 shows an example of an oc-graph and of its inclusion digraph. Similarly to the definition of c-graphs, H describes the inclusions among the clusters: $V(\mu) \subset V(\nu)$ if there exists a directed path from ν to μ in H. Also, $G(\mu)$ denotes the subgraph of G induced by the cluster represented by μ. If μ and ν are two distinct nodes of H such that $V(\mu) \not\subset V(\nu)$, $V(\nu) \not\subset V(\mu)$ and $V' = V(\mu) \cap V(\nu) \neq \emptyset$, then we call V' the *overlap of* $V(\mu)$ *and* $V(\nu)$, and we say that $V(\mu), V(\nu)$ are two *overlapping clusters*. An oc-graph W is said to be *c-connected* if for each non-sink node μ of H, $G(\mu)$ is connected. For example, the oc-graph in Figure 3(a) is c-connected. Observe that c-graphs can be considered as special cases of oc-graphs, because a c-graph $C = (G, T)$ is an oc-graph with no overlap where all arcs of T are oriented downward, from the root to the leaves.

We remark that the class of *intersecting clustered graphs* defined by Omote and Sugiyama [17, 18] is a proper subclass of the overlapping clustered graphs studied in this paper, because in an intersecting clustered graph the subgraph induced by an overlap cannot be further decomposed into clusters. In other words, the inclusion digraph for an intersecting clustered graph is a rooted tree except that it allows sharing of leaves between clusters. For example, the clustering structure depicted in Figure 3 does not satisfy the definition of intersecting clustered graph because nodes ν_2 and μ_2 share node δ_2.

In their work, Feng et al. [9] define the concept of a c-planar drawing of a c-graph. We extend this definition to oc-graphs. An *oc-planar drawing* of an oc-graph $W = (G, H)$ is a representation of W in the plane such that each vertex of G is drawn as a distinct point in the plane, each edge of G is drawn as a simple Jordan curve, and each node μ of H is drawn as a simple closed region $R(\mu)$ according to the following rules. We denote as v a vertex of G and as $p(v)$ the point representing v in the drawing.

R1: $R(\mu)$ contains the drawing of $G(\mu)$.

R2: If $V(\mu) \subset V(\nu)$ then $R(\mu) \subset R(\nu)$, and if the boundaries of $R(\nu)$ and $R(\mu)$ intersect then every connected region of $R(\nu) \cap R(\mu)$ contains at least one vertex of $V(\mu) \cap V(\nu)$.

R3: If $v \notin V(\mu)$, $p(v)$ is outside $R(\mu)$.

R4: There is no *edge crossing*, i.e., any two edges of G never cross.

R5: There is no *edge-region crossing*, i.e., there is no edge of G that crosses the boundary of a region $R(\mu)$ twice.

An oc-graph is *oc-planar* if it admits an oc-planar drawing; for example, the oc-graph of Figure 3 is oc-planar. For the special case that an oc-graph W is a c-graph, a drawing of W that satisfies Rules R1–R5 is called *c-planar*; a c-graph is *c-planar* if it admits a c-planar drawing. For example, the c-graph of Figure 2 is c-planar. Note that, by Rule R1, in an oc-planar drawing the boundaries of two regions $R(\mu), R(\nu)$ necessarily intersect if $V(\mu), V(\nu)$ are overlapping clusters. Conversely, in a c-planar drawing the boundaries of any two regions can be always “shrank” so that they never intersect.

We conclude this section with definitions that are going to be used in the remainder of the paper. Let $C = (G, T)$ be a c-planar graph. We denote as $\Gamma(C)$ a c-planar drawing of C and as $\Gamma(G)$ the planar drawing of G in $\Gamma(C)$. Also, $\Gamma(G(\mu))$ denotes the drawing of $G(\mu)$ in $\Gamma(C)$, for each cluster μ. Analogously, let $W = (G, H)$ be an oc-planar graph. We denote as $\Gamma(W)$ an oc-planar drawing of W, and as $\Gamma(G)$ and $\Gamma(G(\mu))$ the drawing of G and $G(\mu)$ in $\Gamma(W)$, respectively.

A planar drawing $\Gamma(G)$ of a graph G subdivides the plane into topologically connected regions, called *faces*; exactly one of this faces is unbounded, and it is called the *external face*; the other faces are *internal faces.* An internal (resp. external) face f is described by the clockwise (resp. counterclockwise) sequence of vertices and edges that form its boundary. The description of a set of faces for G is *a planar embedding* of G and *the planar embedding* of $\Gamma(G)$; we recall that if a graph is planar and 3-connected, any two planar embeddings of the graph can only differ for their external face [12]. Throughout the paper a face f is regarded as an open set; therefore when we say that a vertex v is *in* f we mean that v lies inside the region f but not on its boundary.

2.2 Characterizing Overlapping Cluster Planarity

Feng et al. [9] gave a characterization of those c-connected clustered graphs $C = (G, T)$ that are c-planar. Their characterization is based on the existence of a planar embedding of G with certain properties.

Theorem 1 [9] *A c-connected c-graph $C = (G, T)$ is c-planar if and only if G admits a planar embedding such that, for each node μ of T, all vertices of $G - G(\mu)$ are in the external face of $G(\mu)$.*

The next theorem can be proved by extending the technique of [9] in order to deal with the more complex structure of an inclusion digraph instead of the structure of an inclusion tree.

Theorem 2 *A c-connected oc-graph $W = (G, H)$ is oc-planar if and only if G admits a planar embedding such that, for each node μ of H, all vertices of $G - G(\mu)$ are in the external face of $G(\mu)$.*

Proof: Suppose that W is oc-planar and let $\Gamma(W)$ be an oc-planar drawing of W. Let μ be a node of H and let v be a vertex of $G - G(\mu)$. Suppose by contradiction that v lies in an internal face of $G(\mu)$. Since by Rule R1, region $R(\mu)$ contains the drawing of $G(\mu)$ in $\Gamma(W)$, it follows that also the point $p(v)$ representing v is inside $R(\mu)$, which however contradicts Rule R3. Therefore v is in the external face of $G(\mu)$.

Conversely, suppose that G has a planar embedding that satisfies the statement. Denote by ψ such an embedding. To prove that W is oc-planar, compute a planar drawing $\Gamma(G)$ of G that preserves ψ, i.e., a planar drawing that induces the set of faces of ψ (this can be done by applying standard graph drawing algorithms [7]). Incrementally construct the cluster regions by following a suitable order. Namely, let $S = \mu_1, \ldots, \mu_h$ be the sequence of the non-sink nodes of H ordered according to the reverse of a topological sorting. This implies that μ_h is the source of H and that for each μ_j ($1 \leq j \leq h$) those non-sink nodes of H reachable from μ_j with a directed path have index smaller than j.

$\Gamma(W)$ is constructed from $\Gamma(G)$ by executing $h-1$ steps. At Step j ($1 \leq j \leq h-1$)), region $R(\mu_j)$ is added to the drawing. The computed drawing, denoted as $\Gamma^{(j)}(W)$ is $\Gamma(G) \cup R(\mu_1), \cup \ldots \cup R(\mu_j)$ and represents G together with clusters $V(\mu_1), \ldots, V(\mu_j)$. We describe the construction and prove by induction that $\Gamma^{(j)}(W)$ verifies Rules R1-R5, limited to the regions $R(\mu_1), \ldots R(\mu_j)$.

At Step 1, region $R(\mu_1)$ is defined. The boundary of $R(\mu_1)$ is a simple closed curve, denoted as $\mathcal{C}$, drawn in the external face of $\Gamma(G(\mu_1))$. Curve $\mathcal{C}$ follows the profile of the boundary of the external face of $\Gamma(G(\mu_1))$, $\epsilon > 0$ distance away (on the outside) from it. Since $G(\mu_1)$ is connected, $R(\mu_1)$ contains drawing $\Gamma(G(\mu_1))$ and therefore $\Gamma^{(1)}(W)$ satisfies Rule R1. $\Gamma^{(1)}(W)$ contains the region of exactly one cluster and therefore Rule R2 is trivially satisfied. Defining $\Gamma^{(1)}(W)$ we choose ϵ to be sufficiently small such that $\mathcal{C}$ crosses each edge incident to $G(\mu_1)$ exactly once and that it does not cross any other vertex or edge of the drawing of $G - G(\mu_1)$. The existence of such an ϵ follows from the fact that $\Gamma(G)$ preserves ψ which, by assumption, has all vertices of $G - G(\mu_1)$ (and hence also all edges of $G - G(\mu_1)$) in the external face of $G(\mu_1)$. Such a choice of ϵ guarantees that $\Gamma^{(1)}(W)$ satisfies Rule R3 and Rule R5. Finally, Rule R4 is satisfied by the planarity of $\Gamma(G)$.

Suppose by induction that $\Gamma^{(j-1)}(W)$ ($j > 2$) verifies Rules R1-R5, limited to regions $R(\mu_1)$, …,$R(\mu_{j-1})$ and execute Step j. If μ_j has some outgoing edges that are incident to non-sink nodes, denote as $\nu_1, \ldots, \nu_k$ such non-sink nodes. Note that for the chosen ordering, regions $R(\nu_1), \ldots, R(\nu_k)$ have been already drawn in $\Gamma^{(j-1)}(W)$. Consider the drawing Γ' given by the union of $\Gamma(G(\mu_j))$ with the boundaries of regions $R(\nu_1), \ldots, R(\nu_k)$. If μ_j does not have outgoing edges incident to non-sink nodes, Γ' coincides with $\Gamma(G(\mu_j))$. See also Figure 4 for an illustration of the construction of Γ'. The boundary of $R(\mu_j)$ is a simple closed curve, denoted as $\mathcal{C}$, that follows the profile of the external boundary of Γ' $\epsilon > 0$ distance away (on the outside) from it. Since by construction $R(\mu_j)$ contains $\Gamma(G(\mu_j))$, $\Gamma^{(j)}(W)$ satisfies Rule R1. Also, since by hypothesis all the vertices that do not belong to $G(\mu_j)$ are in the external face of $G(\mu_j)$, then the choice of ϵ can be such that both R3 and R5 are satisfied; Rule R4 is guaranteed by the planarity of $\Gamma(G)$. It remains to prove Rule R2. By construction, the regions of all clusters contained in $V(\mu_j)$ are properly contained in $R(\mu_j)$. Also, by the inductive hypothesis and since all vertices and edges that do not belong to $G(\mu_j)$ are in the external face of $G(\mu_j)$, ϵ can be further reduced in order to avoid intersections between $R(\mu_j)$ and regions of clusters that do not overlap with $V(\mu_j)$. Finally, suppose that $V(\mu_i)$ is a cluster that overlaps with $V(\mu_j)$ ($i \leq j$), and let R be a connected region of $R(\mu_j) \cap R(\mu_i)$. Since the boundary of $R(\mu_i)$ has been constructed by following the profile of $\Gamma(G(\mu_i))$, then there must be an edge e of $G(\mu_i)$ that enters inside R and that crosses the boundary of $R(\mu_j)$. Since e cannot crosses twice the boundary of $R(\mu_j)$ (otherwise Rule R5 would be violated), it follows that there is an end-vertex of e inside R. Hence, Rule R2 is satisfied. □

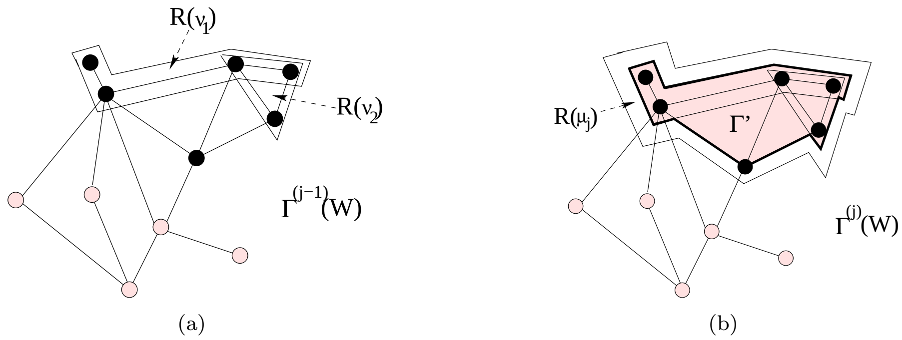


Figure 4: Illustration of Step j in the proof of Theorem 2. (a) Drawing $\Gamma^{(j-1)}(W)$; black vertices belong to cluster $V(\mu_j)$, which contains sub-clusters $V(\nu_1)$ and $V(\nu_2)$. (b) Drawing $\Gamma^{(j)}(W)$; region R_{μ_j} is constructed by following the boundary of Γ' (bold polygon).

In Feng et al. [9], the characterization of Theorem 1 is used to design an $O(n^2)$ time algorithm to test whether C is c-planar, and in the positive case to compute a c-planar drawing of C. Their approach looks for a planar embedding of G that verifies the properties given in Theorem 1; the algorithm proceeds bottom-up, with a post-order visit of T. One of the key-ideas is that for each node μ of T $G(\mu)$ can be tested independently of any other cluster that is not in the subtree rooted at μ. Unfortunately, it does not seem immediate to follow a similar approach to design an algorithm for oc-planarity directly based on Theorem 2, mainly because the graphs induced by the clusters of an oc-graph cannot be always tested independently of each other due to their overlaps. This observation motivates us to better understand the circumstances under which an oc-planarity testing algorithm can be designed by using a corresponding algorithm for c-planarity. The next definitions and lemma will be of use for this purpose.

Let $W = (G, H)$ be a c-connected oc-graph. Given any two overlapping clusters $V(\mu)$ and $V(\nu)$, let V' denote their overlap. We say that:

- W is *1-oc-connected* if for every pair $V(\mu), V(\nu)$ of overlapping clusters, V' induces a connected subgraph of G.
- W is *2-oc-connected* if it is 1-oc-connected and for every pair $V(\mu), V(\nu)$ of overlapping clusters, at least one of $V(\mu) - V'$ and $V(\nu) - V'$ induces a connected subgraph of G.
- W is *3-oc-connected* if it is 1-oc-connected and for every pair $V(\mu), V(\nu)$ of overlapping clusters, both $V(\mu) - V'$ and $V(\nu) - V'$ induce connected subgraphs of G.

Observe that if W is 3-oc-connected, it is also 2-oc-connected. For example, Figure 5 shows the drawings of a 1-oc-connected, a 2-oc-connected, and a 3-connected oc-graph.

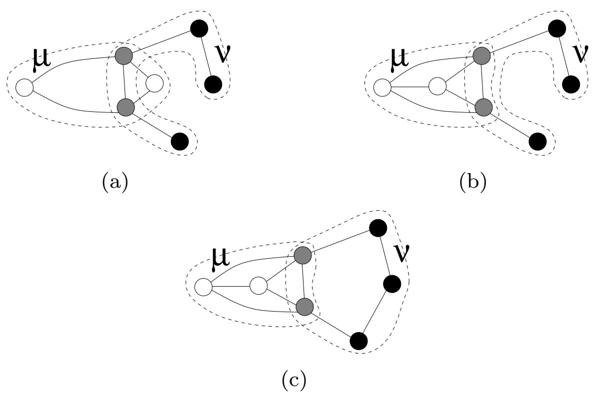


Figure 5: Three oc-planar drawings of three distinct oc-graphs; in each oc-graph, white and gray vertices belong to a cluster $V(\mu)$ while black and gray vertices belong to a cluster $V(\nu)$. The cluster regions have dashed boundaries. (a) The oc-graph is 1-oc-connected. (b) The oc-graph is 2-oc-connected. (c) The oc-graph is 3-oc-connected. Observe that in the drawings (b) and (c), the boundaries of the cluster regions of μ and ν share exactly two points, while this is not true for drawing (a).

Lemma 1 *Let $W = (G, H)$ be a 2-oc-connected oc-graph. If W is oc-planar, there exists an oc-planar drawing of W such that the boundaries of the regions of any two overlapping clusters share exactly two points.*

Proof: Construct an oc-planar drawing $\Gamma(W)$ of W by applying the procedure described in the proof of Theorem 2. Let $V(\mu)$ and $V(\nu)$ be two overlapping clusters of W and let V' denote their overlap. By hypothesis, W is 2-oc-connected. This implies that V' induces a connected subgraph of G and that at least one of $V(\mu) - V'$ and $V(\nu) - V'$ induces a connected subgraph of G. Without loss of generality, assume that the subgraph induced by $V(\mu) - V'$ is connected. Let $R(\mu)$, $R(\nu)$ be the cluster regions of μ and ν, respectively, and let R' denote the region containing the subgraph induced by V' and delimited by the boundaries of $R(\mu)$ and $R(\nu)$. Clearly, the number of intersections between the boundaries of $R(\mu)$ and $R(\nu)$ is an even number. Also, since $\Gamma(W)$ is oc-planar and since $R(\mu)$ and $R(\nu)$ have been constructed by following the profiles of μ and ν, respectively, it follows that R' is simple and connected (because V' induces a connected subgraph) and $R(\mu) - R'$ is simple and connected (because $V(\mu) - V'$ induces a connected subgraph). Therefore, the boundaries of $R(\mu)$ and $R(\nu)$ intersect in exactly two points (see also Figure 5 for an example). $\square$

3 Models and Algorithms for Overlapping Clustered Graphs

In this section we study first the simple case of an oc-graph having exactly two overlapping clusters (Subsection 3.1) and then extend the investigation to a meaningful class of oc-graphs with many clusters (Subsection 3.2).

3.1 A Model for Two Clusters

Let $W = (G, H)$ be a c-connected oc-graph where the vertices of G are grouped in exactly two overlapping clusters $V(\mu)$ and $V(\nu)$. We call W a *two overlapping clustered graph* or *toc-graph* for short. An example of a toc-graph is in Figure 6(a). Let $C = (G, T)$ be the c-graph constructed by considering three disjoint clusters $V(\mu) - V'$, V', and $V(\nu) - V'$, where V' is the overlap of $V(\mu)$ and $V(\nu)$. In $C = (G, T)$, the leaves of T are the vertices of G; T has four internal nodes: The root r and three children of r, denoted as α, β, and γ, where $V(\alpha) = V(\mu) - V'$, $V(\beta) = V(\nu) - V'$, and $V(\gamma) = V'$. We call C the *c-image of* W. Figure 6(b) shows the c-image of the toc-graph in Figure 6(a).

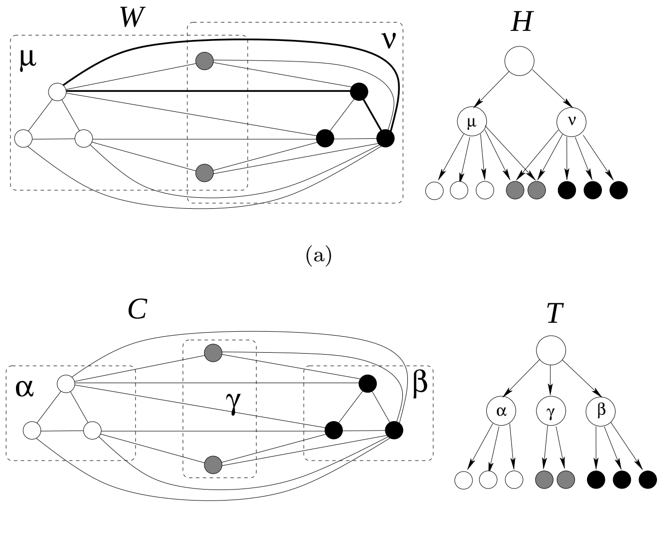


Figure 6: (a) A toc-graph W. Gray vertices belong to the overlap. (b) The c-image C of W.

A natural question to ask is whether testing a toc-graph for oc-planarity is equivalent to testing its c-image for c-planarity. Figure 7(a) shows a drawing of a toc-graph W and Figure 7(b) shows a c-planar drawing of the c-image C of W where each of the clusters of C induces a connected subgraph of G.

Notice however that the planar embedding of the c-planar drawing of Figure 7(b) cannot be the embedding of an oc-planar drawing of W; namely, either Rule R1 is violated because some of the edges of $G(\mu)$ are not contained in $R(\mu)$ (see Figure 7(a)) or $R(\mu)$ must contain $R(\nu)$ which violates Rule R3.

On the positive side, notice however that the planar embedding can be changed by suitably redefining the external face, in order to guarantee both the existence of an oc-planar drawing of W and the existence of a c-planar drawing of C; for example, one can choose face f of Figure 7(b) as the new external face, as shown in Figure 7(c).

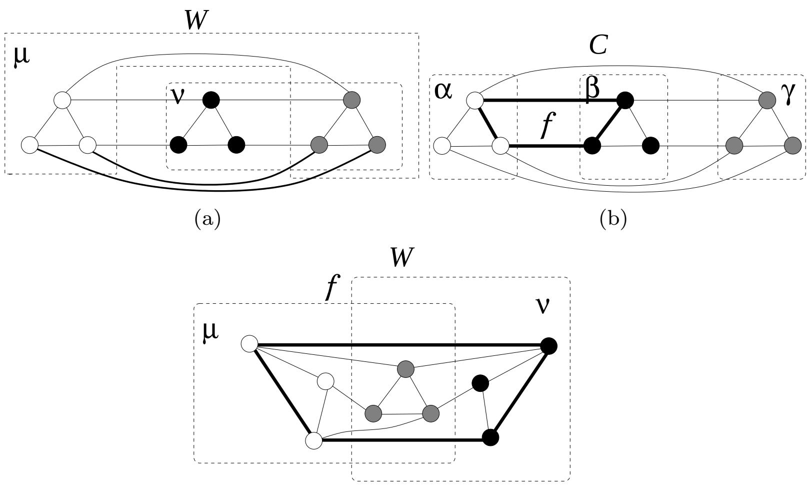


Figure 7: (a) A toc-graph W; the drawing depicted in the figure is not oc-planar, since the two edges in bold are not completely inside the region of μ; an oc-planar drawing of W with this embedding cannot exist. (b) A c-planar drawing of the c-image C of W. (c) An oc-planar drawing of W obtained by changing its original embedding; namely, face f is chosen as the new external face.

Lemma 2 *Let $W = (G, H)$ be a toc-graph and let C be the c-image of W. If C is c-planar, then W is oc-planar.*

Proof: Let $V(\mu)$, $V(\nu)$ be the two overlapping clusters of W and let V' be the overlap of $V(\mu)$, $V(\nu)$. Let the clusters of C be $V(\alpha) = V(\mu) - V'$, $V(\beta) = V(\nu) - V'$, and $V(\gamma) = V'$. Let $\Gamma(C)$ be a c-planar drawing of C and let ϕ be the planar embedding of $\Gamma(G)$. We define a planar embedding ψ of G that satisfies the statement of Theorem 2, which implies that W is oc-planar.

If the boundary of the external face of ϕ contains both a vertex of $V(\alpha)$ and a vertex of $V(\beta)$, then ϕ coincides with ψ. Otherwise, ψ is obtained from

ϕ by choosing a new external face whose boundary contains a vertex of $V(\alpha)$ and a vertex of $V(\beta)$. We claim that such a face always exists in ϕ. Indeed, if such a face did not exist, for every pair of vertices a, b such that $a \in V(\alpha)$ and $b \in V(\beta)$, there would exist a simple cycle χ consisting only of vertices of $V(\gamma)$ such that a and b lie one inside and one outside χ. However, this would imply that the region of cluster $V(\gamma)$ in $\Gamma(C)$ contains a vertex that does not belong to $V(\gamma)$, violating Rule R3 and therefore contradicting the fact that $\Gamma(C)$ is a c-planar drawing. This proves the claim.

Let f be a face of ϕ with both a vertex a of $V(\alpha)$ and a vertex b of $V(\beta)$ on its boundary; let ψ be the planar embedding obtained from ϕ by choosing f as the external face. In order to prove that ψ satisfies the statement of Theorem 2, we show that every vertex of $V(\alpha)$ is in the external face of $G(\nu)$ and that every vertex of $V(\beta)$ is in the external face of $G(\mu)$. Clearly, vertex a is in the external face of $G(\nu)$ because it is a vertex of the boundary of the external face of ψ. Let a' be any vertex of $V(\alpha)$ different from a and assume for a contradiction that a' is not in the external face of $G(\nu)$. This implies that there exists a cycle χ in ψ consisting only of vertices and edges of $G(\nu)$ and such that a' is in the interior of χ while a is in the exterior of χ. Since embeddings ϕ and ψ can only differ for their external faces, we have that in $\Gamma(C)$ cycle χ has one of the vertices a and a' in its interior, while the other one in its exterior. Since a, a' both belong to $V(\alpha)$, the boundary of region $R(\alpha)$ and χ cross each other twice. If these two crossings are between one edge of χ and $R(\alpha)$, then $\Gamma(C)$ violates Rule R5. Otherwise, there must be a vertex of χ inside $R(\alpha)$, which violates Rule R3 because no vertex of χ belongs to $V(\alpha)$. Hence, cycle χ cannot exist and a' is in the external face of $G(\nu)$ in ψ.

By a symmetric argument, every vertex of $V(\beta)$ is in the external face of $G(\mu)$. Therefore ψ satisfies the statement of Theorem 2 and W is oc-planar. □

As the next lemmas show, it is however not always true that the oc-planarity of a toc-graph implies the c-planarity of its c-image.

Lemma 3 *There exists an oc-planar toc-graph whose c-image is not c-planar.*

Proof: Let $W = (G, H)$ be the toc-graph of Figure 6(a), where $V(\mu)$ is the set of white and gray vertices while $V(\nu)$ is the set of black and gray vertices. Let C be the c-image of W (Figure 6(b)); the three clusters of C are the gray vertices, the black vertices, and the white vertices. As the figure shows, W is oc-planar because it has an oc-planar drawing. The bold edges in the figure form a cycle, denoted as χ, consisting of only black and white vertices and such that a gray vertex is inside χ and the other gray vertex is outside χ. Since G is 3-connected, cycle χ leaves the two gray vertices one inside and one outside in all planar embeddings of G. As a consequence, every clustered drawing of C such that the two gray vertices are inside region $R(\gamma)$ must contain two crossings between χ and the boundary of $R(\gamma)$. If these two crossings are between one edge of χ and $R(\gamma)$, then Rule R5 is violated. Otherwise, there must be a vertex of χ inside $R(\gamma)$ which violates Rule R3 because no vertex of χ belongs to $V(\gamma)$. It follows that C is not c-planar. □

The counterexample of Lemma 3 exploits the fact that the vertices of the overlap induce a non-connected subgraph of G and thus the toc-graph is not 1-oc-connected. One can ask whether the converse of Lemma 2 holds for 1-oc-connected toc-graphs. Unfortunately, also in this case the answer is negative.

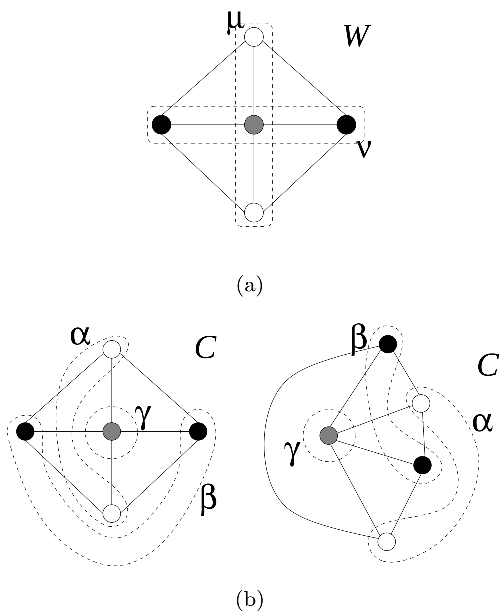


Figure 8: (a) An oc-planar drawing of a toc-graph $W = (G, H)$. (b) Two drawings of the c-image C of W for two different planar embeddings of G. Both the drawings are not c-planar.

Lemma 4 *There exist oc-planar 1-oc-connected toc-graphs whose c-image is not c-planar.*

Proof: Consider the toc-graph $W = (G, H)$ drawn in Figure 8(a). As the figure shows, W admits an oc-planar drawing. We show however that the c-image C of W is not c-planar. Since G is a 3-connected graph, two distinct planar embeddings of G differ only for their external face. By the symmetry of G, it is sufficient to consider the two possible classes of planar embeddings of G depicted in Figure 8(b). For each of these two embeddings, any drawing of the simple closed regions $R(\alpha)$ and $R(\beta)$ is such that either these regions intersect or the boundary of one of them crosses twice some edge of G. Hence, a c-planar drawing of C does not exist. □

The next lemma shows that for the family of 2-oc-connected toc-graphs the oc-planarity implies the c-planarity of the c-image.

Lemma 5 *Let $W = (G, H)$ be a 2-oc-connected toc-graph and let C be the c-image of W. If W is oc-planar then C is c-planar.*

Proof: Let $V(\mu)$, $V(\nu)$ be the two overlapping clusters of W and let V' be the overlap of $V(\mu)$, $V(\nu)$. Let the clusters of C be $V(\alpha) = V(\mu) - V'$, $V(\beta) = V(\nu) - V'$, and $V(\gamma) = V'$.

Since W is oc-planar and 2-oc-connected, by Lemma 1 W has an oc-planar drawing $\Gamma(W)$ such that the boundaries of the regions $R(\mu)$ and $R(\nu)$ intersect in exactly two points, which we denote as p_1 and p_2 (refer also to Figure 9 for an illustration).

To prove that C is c-planar, we construct a c-planar drawing $\Gamma'(C)$ of C from $\Gamma(W)$. Namely, in $\Gamma'(C)$ the drawing of G is the same as in $\Gamma(W)$. Regions $R(\alpha)$, $R(\beta)$, and $R(\gamma)$ are defined as follows. Denote as $\mathcal{C}(\mu)$ the boundary of $R(\mu)$ and as $\mathcal{C}(\nu)$ the boundary of $R(\nu)$ in $\Gamma(W)$. Points p_1 and p_2 split $\mathcal{C}(\mu)$ into two distinct curves, denoted as $\mathcal{C}'(\mu), \mathcal{C}''(\mu)$ and having p_1 and p_2 as their end-points; $\mathcal{C}'(\mu)$ is outside $R(\nu)$ while $\mathcal{C}''(\mu)$ is inside $R(\nu)$. Similarly, $\mathcal{C}'(\nu)$ ($\mathcal{C}''(\nu)$) denotes the portion of $\mathcal{C}(\nu)$ between p_1 and p_2 outside (inside) $R(\mu)$. In $\Gamma(C)$, the boundary of $R(\alpha)$ is the union of $\mathcal{C}'(\mu)$ and $\mathcal{C}''(\nu)$; the boundary of $R(\beta)$ is the union of $\mathcal{C}'(\nu)$ and $\mathcal{C}''(\mu)$; the boundary of $R(\gamma)$ is a simple closed curve that follows the profile of $\mathcal{C}''(\mu) \cup \mathcal{C}''(\nu)$, $\epsilon > 0$ distance away (on the inside) from it. Observe that, with the construction described so far, the boundary of $R(\alpha)$ and $R(\beta)$ share exactly the points p_1 and p_2; to remove these two contact points between the two regions, we can always slightly move the corners of $R(\alpha)$ and $R(\beta)$ at points p_1 and p_2 by a suitable $\epsilon > 0$ distance.

We finally show that $\Gamma'(C)$ verifies Rules R1-R5. Rules R1 and R2 are satisfied by construction. Rule R3 follows from Rules R1 and R2 and the fact that clusters α, β, and γ are at the same level of the inclusion tree of C and that every vertex of G belongs to exactly one of α, β, and γ. Rule R4 is a consequence of the planarity of $\Gamma(G)$. Finally, observe that the boundary of each cluster region $R(\xi)$ ($\xi \in \{\alpha, \beta, \gamma\}$) can be made sufficiently close to the drawing $\Gamma(G(\xi))$ of $G(\xi)$ such that the region crosses each edge incident to $G(\xi)$ exactly once. Rule R5 is thus satisfied. □

Lemmas 2 and 5 can be summarized as follows.

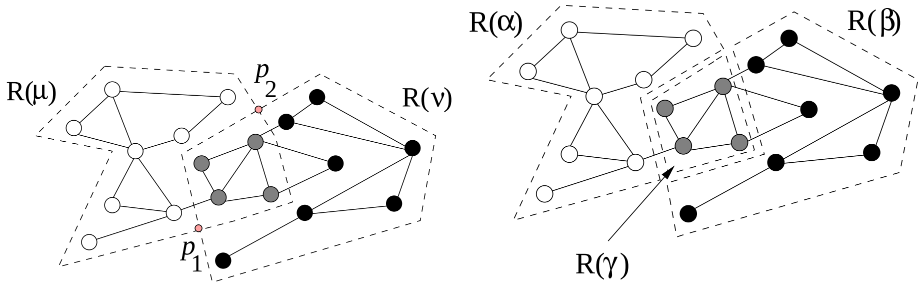
Theorem 3 *Let $W = (G, H)$ be a 2-oc-connected toc-graph and let C be the c-image of W. W is oc-planar if and only if C is c-planar.*

Based on Theorem 3 and on known results about c-planarity testing ([4, 5, 2, 11]), one can design polynomial-time algorithms for oc-planarity testing. The following result summarizes the algorithmic contribution of this section.

Theorem 4 *Let $W = (G, H)$ be a toc-graph and let n be the number of vertices of G. The following statements hold:*

(a) If W is 2-oc-connected then it can be tested for oc-planarity in $O(n^2)$ time.

(b) If W is 3-oc-connected then it can be tested for oc-planarity in $O(n)$ time.



(a) (b)

Figure 9: Illustration of the procedure described in Lemma 5 to modify an oc-planar drawing of a toc-graph W in order to get a c-planar drawing of the c-image of W.

Proof: Let $C = (G, T)$ be the c-image of $W = (G, H)$. C can be constructed from W in $O(n)$ time. Indeed, C and W have the same underlying graph G, and the inclusion tree T of C can be constructed from the inclusion digraph H be simply considering a visit of H. Both in Statements (a) and (b), the overlap induces a connected subgraph and hence, by Theorem 3, testing the oc-planarity of W is equivalent to testing the c-planarity of C.

To prove Statement (a), we recall that in [11], the class of *almost c-connected* c-graphs is defined, and an $O(n^2)$-time c-planarity testing algorithm is described for this class of c-graphs. A c-graph is almost c-connected if one of the following conditions applies:

- All clusters inducing non-connected subgraphs of G lie on a same path from the root to a leaf of T.
- For each node μ of T such that $G(\mu)$ is not connected, the cluster of the parent of μ induces a connected subgraph of G, and the cluster of every sibling of μ induces a connected subgraph of G.

In the hypothesis of Statement (a), W is 2-oc-connected, and therefore at most one cluster of C – say $V(\alpha)$ – induces a non-connected subgraph of G, and both G, $G(\beta)$ and $G(\gamma)$ are connected. This implies that C is an almost c-connected c-graph, and can be tested for c-planarity in $O(n^2)$ time.

In the hypothesis of Statement (b), W is 3-oc-connected, and therefore C is a c-connected c-graph. Hence, the c-planarity of C can be tested in $O(n)$ time with the algorithm of Dahlhaus et al. [4, 5] or with the algorithm of Cortese et al. [2]. □

3.2 A Model for Many Clusters

A natural extension of the study in the previous section is to consider oc-graphs with $m \geq 2$ clusters and such that only pairs of clusters can overlap.

A *multiple-two overlapping clustered graph*, also called *moc-graph* for short, is an oc-graph such that each cluster can overlap with at most another cluster. Notice that a moc-graph can have clusters included in other clusters and that a toc-graph is a special case of moc-graph. The oc-graph depicted in Figure 3(a) is an example of moc-graph.

The *c-image* of a moc-graph $W = (G, H)$ is the c-graph $C = (G, T)$ defined as follows:

- Each non-overlapping cluster of W is also a cluster of C;
- For each pair $V(\mu), V(\nu)$ of overlapping clusters of W, let V' be their overlap; $V(\mu)$, $V(\nu)$, and V' define the following four clusters in C: $V(\alpha) = V(\mu) - V'$, $V(\beta) = V(\nu) - V'$, $V(\gamma) = V'$, and $V(\lambda) = V(\mu) \cup V(\nu)$.

The c-graph depicted in Figure 2 is the c-image of the moc-graph of Figure 3. Observe that, in the special case all vertices of a moc-graph W are grouped into exactly two overlapping clusters $V(\mu)$ and $V(\nu)$ (i.e., W is a toc-graph), then the c-image of W coincides with the one defined for a toc-graph, where λ is the root of T. From this observation, it immediately follows that the counterexamples of Lemmas 3 and 4 also hold for moc-graphs, because a toc-graph and its c-image can be considered as a special case of a moc-graph and its c-image. The next lemma extends Lemma 5 to the case of many clusters.

Lemma 6 *Let $W = (G, H)$ be a 2-oc-connected moc-graph and let C be the c-image of W. If W is oc-planar then C is c-planar.*

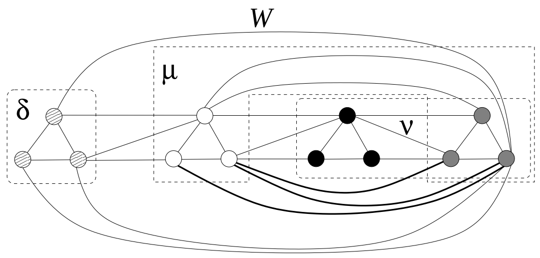
Proof: If W is oc-planar, the c-planarity of C can be shown by a similar argument as in the proof of Lemma 5. Namely, based on Lemma 1, we can construct an oc-planar drawing $\Gamma(W)$ such that, for every pair of overlapping clusters $V(\mu), V(\nu)$, the boundaries of the regions $R(\mu)$, $R(\nu)$ intersect in exactly two points p_1 and p_2 (see Figure 9(a)). To prove that C is c-planar, we show how to construct a c-planar drawing $\Gamma'(C)$ of C from $\Gamma(W)$. The drawing of G in $\Gamma'(C)$ is the same as in $\Gamma(W)$. Also, for each non-overlapping cluster $V(\delta)$, the drawing of the region $R(\delta)$ is the same as in $\Gamma(W)$. For every pair $V(\mu), V(\nu)$ of overlapping clusters with overlap V', regions $R(\alpha)$, $R(\beta)$, and $R(\gamma)$ are defined as follows. Denote as $\mathcal{C}(\mu)$ the boundary of $R(\mu)$ and as $\mathcal{C}(\nu)$ the boundary of $R(\nu)$ in $\Gamma(W)$. Points p_1 and p_2 split $\mathcal{C}(\mu)$ into two curves, denoted as $\mathcal{C}'(\mu), \mathcal{C}''(\mu)$ and having p_1 and p_2 as their end-points; $\mathcal{C}'(\mu)$ is outside $R(\nu)$ while $\mathcal{C}''(\mu)$ is inside $R(\nu)$. Similarly, $\mathcal{C}'(\nu)$ ($\mathcal{C}''(\nu)$) denotes the portion of $\mathcal{C}(\nu)$ between p_1 and p_2 outside (inside) $R(\mu)$. In $\Gamma(C)$, the boundary of $R(\alpha)$ is the union of $\mathcal{C}'(\mu)$ and $\mathcal{C}''(\nu)$; the boundary of $R(\beta)$ is the union of $\mathcal{C}'(\nu)$ and $\mathcal{C}''(\mu)$; the boundary of $R(\gamma)$ is a simple closed curve that follows the profile of $\mathcal{C}''(\mu) \cup \mathcal{C}''(\nu)$, $\epsilon > 0$ distance away (on the inside) from it. Observe that, with the construction described so far, the boundary of $R(\alpha)$ and $R(\beta)$ exactly share the two points p_1 and p_2; to remove these two contact points between the two regions, we can always slightly move the corners of $R(\alpha)$ and $R(\beta)$ at points p_1 and p_2 by a suitable $\epsilon > 0$ distance. Now, consider the region $R(\mu) \cup R(\nu)$ in $\Gamma(W)$ and let Λ be its boundary. The boundary of the region of $V(\mu) \cup V(\nu)$ in $\Gamma'(C)$ is drawn as a simple closed curve that follows Λ, $\epsilon > 0$ distance away (on the outside) from it.

We prove that $\Gamma'(C)$ verifies Rules R1-R5. Rule R1 and R2 are verified by construction. Indeed, every non-overlapping cluster region satisfies Rule R1 in $\Gamma'(C)$, since it is the same region as in $\Gamma(W)$ and $\Gamma(W)$ is an oc-planar drawing. Also, for every pair of overlapping clusters $V(\mu), V(\nu)$, we can draw $R(\alpha), R(\beta), R(\gamma)$ and $R(\lambda)$ in such a way that they contain the drawing of the subgraph induced by $V(\alpha), V(\beta), V(\gamma)$ and $V(\delta)$, respectively. By suitably choosing the boundaries of the cluster regions, we can keep in the exterior of $R(\delta)$ all regions of clusters non included in any of $V(\alpha), V(\beta)$ and $V(\gamma)$. Likewise, every cluster possibly included in $V(\alpha), V(\beta)$ or $V(\gamma)$ is in the interior of $R(\alpha), R(\beta)$ or $R(\gamma)$. Every non-overlapping cluster region $R(\delta)$ in $\Gamma(W)$ satisfies Rule R3 in $\Gamma'(C)$, because the same rule is satisfied in $\Gamma(W)$. Consider now a pair of overlapping clusters $V(\mu)$ and $V(\nu)$. Since $\Gamma(W)$ is an oc-planar drawing, all vertices of $V-(V(\mu)\cup V(\nu))$ are outside $R(\mu)$ and outside $R(\nu)$. We can always choose distance ϵ sufficiently small for $R(\lambda)$ to guarantee they are also outside $R(\lambda)$ in $\Gamma'(C)$. Moreover, all vertices of $V(\mu) - V'$ are outside $R(\nu)$ in $\Gamma(W)$ and by construction $R(\alpha)$ does not contain vertices that are not in $V(\alpha)$. Analogously, $R(\beta)$ does not contain vertices that are not in $V(\beta)$. Consider a vertex $v \in V(\alpha)$. If $p(v)$ were inside $R(\gamma)$ in $\Gamma'(C)$, by the connectivity of $G(\gamma)$ and by construction of $R(\gamma)$, $p(v)$ would be inside $R(\nu)$ in $\Gamma(W)$, a contradiction. With a symmetric argument, every vertex of $V(\beta)$ is drawn outside $R(\gamma)$ in $\Gamma'(C)$. It follows that Rule R3 is satisfied. Rule R4 is satisfied as a consequence of the planarity of $\Gamma(G)$. Finally, observe that, for each pair of overlapping clusters $V(\mu), V(\nu)$ in $\Gamma(W)$, the boundary of each cluster region $R(\xi)$ ($\xi \in \{\alpha, \beta, \gamma\}$), can be made sufficiently close to $G(\xi)$ in such a way that it crosses each edge incident to $G(\xi)$ exactly once. Rule R5 is therefore satisfied. □

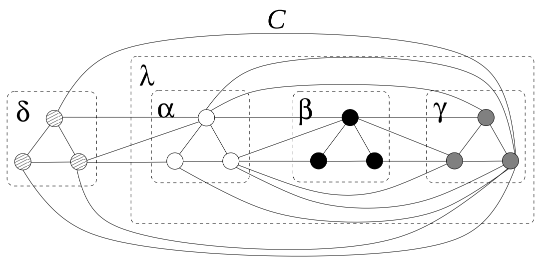
Unfortunately, Lemma 2 cannot be extended to moc-graphs, even for very simple types of moc-graphs consisting of just three clusters.

Lemma 7 *There exists a moc-graph $W = (G, H)$ with exactly three clusters, such that the c-image of W is c-planar but W is not oc-planar.*

Proof: Let $W = (G, H)$ be the moc-graph of Figure 10(a). White and gray vertices define a cluster $V(\mu)$, and black and gray vertices define a cluster $V(\nu)$ that overlaps with $V(\mu)$. The remaining three vertices define a third cluster $V(\delta)$ that does not overlap with the other clusters. As shown by Figure 10(b), there exists a c-planar drawing of the c-image of W; but W is not oc-planar. Indeed, by Theorem 2, the embedding of every oc-planar drawing of W must have the black vertices and the vertices of $V(\delta)$ in the external face of $G(\mu)$, i.e., outside every cycle of $G(\mu)$. However, the embedding of G in Figures 10(a) and 10(b) has a cycle χ such that the black vertices lie inside χ and the vertices of $V(\delta)$ are outside. Since G is a 3-connected graph, cycle χ (which separates black vertices and vertices of $V(\delta)$) exists in any planar embedding of G, and therefore W does not admit an oc-planar drawing. □



(a)



(b)

Figure 10: (a) A moc-graph W that is not oc-planar; the drawing in the figure is not oc-planar since the three edges in bold should be completely inside the region of μ. (b) A c-planar drawing of the c-image C of W.

Motivated by Lemma 7, we consider a sub-family of moc-graphs where each cluster has an edge connecting its non overlapping portion to some "external vertices". More formally, a moc-graph $W = (G, H)$ is *externally connected* if for each pair $V(\mu)$, $V(\nu)$ of overlapping clusters with overlap V' there are two edges $(u_1, v_1), (u_2, v_2)$, called *bridges*, with $u_1 \in V(\mu) - V'$, $u_2 \in V(\nu) - V'$ and $v_1, v_2 \notin V(\mu) \cup V(\nu)$. For example, Figure 3(a) is an externally connected moc-graph; the moc-graph in Figure 10(a) is not externally connected, because none of the black vertices is the end-vertex of a bridge.

The proof of the following lemma relies on the fact that in every planar embedding of an externally connected moc-graph the bridges force the external face of the subgraph induced by any pair of overlapping clusters $V(\mu)$, $V(\nu)$ to have at least one vertex of $V(\mu) - V'$ and one vertex of $V(\nu) - V'$ on its boundary.

Lemma 8 *Let $W = (G, H)$ be an externally connected moc-graph and let C be the c-image of W. If C is c-planar, then W is oc-planar.*

Proof: Let $C = (G, T)$ be the c-image of W. If C is c-planar, let ψ be the planar embedding of a c-planar drawing $\Gamma(C)$ of C. We show that ψ satisfies the conditions of Theorem 2. Let μ be a node of H and let v be a vertex of $G - G(\mu)$. We need to show that v is in the external face of $G(\mu)$. Two cases are possible:

- $V(\mu)$ does not overlap with another cluster. In this case, μ is also a node of T, and therefore v is in the external face of $G(\mu)$ because ψ is the embedding of a c-planar drawing of C.
- $V(\mu)$ overlaps with another cluster $V(\nu)$. Let V' be the overlap, and let $V(\alpha) = V(\mu) - V'$, $V(\beta) = V(\nu) - V'$, $V(\gamma) = V'$, and $V(\lambda) = V(\mu) \cup V(\nu)$. We distinguish between two subcases:
  - $v \notin V(\beta)$: In this case, $v \notin V(\lambda)$ and hence it is drawn outside the region of cluster $V(\lambda)$ in $\Gamma(C)$. It follows that v is in the external face of $G(\mu)$ in ψ.
  - $v \in V(\beta)$: In this case, suppose by contradiction that there existed in ψ an internal face f of $G(\mu)$ such that v is in f. Since W is externally connected, there exists an edge (u, w) such that $u \in V(\beta)$ and $w \notin V(\lambda)$. Vertex u must lie in the external face of $G(\mu)$, because otherwise there would be a cross between (u, w) and the external boundary of $G(\mu)$, contradicting the fact that ψ is a planar embedding. However, since in $\Gamma(C)$ the cluster region $R(\beta)$ is a simple closed region and contains both u and v, it follows that there would be an edge region crossing between the boundary of $R(\beta)$ and an edge of the boundary of f, a contradiction.

□

Lemmas 8 and 6 can be summarized as follows.

Theorem 5 *Let $W = (G, H)$ be a moc-graph such that W is 2-oc-connected and externally connected. Let C be the c-image of W. W is oc-planar if and only if C is c-planar.*

Combining Theorem 5 with the results in [2, 4, 5, 11] the following result can be proved.

Theorem 6 *Let $W = (G, H)$ be an externally connected moc-graph and let n be the number of vertices of G. The following statements hold:*

(a) If W is 2-oc-connected then it can be tested for oc-planarity in $O(n^2)$ time.

(b) If W is 3-oc-connected then it can be tested for oc-planarity in $O(n)$ time.

Proof: Let $C = (G, T)$ be the c-image of $W = (G, H)$. C can be constructed from W in $O(n)$ time. Indeed, C and W have the same underlying graph G, and the inclusion tree T of C can be constructed by simply considering a bottom-up visit of the inclusion digraph H.

Notice that for each pair of overlapping clusters $V(\mu), V(\nu)$ of W, cluster $V(\mu) \cup V(\nu)$ of C induces a connected subgraph of G, because both $V(\mu)$ and $V(\nu)$ induce connected subgraphs of G and they share at least one vertex.

In the hypothesis of Statement (a) (i.e., if W is 2-oc-connected) C belongs to the class of almost c-connected c-graphs [11], whose definition has been recalled in the proof of Theorem 4. Namely, with the usual notation, let $V(\alpha)$, $V(\beta)$, $V(\gamma)$ and $V(\lambda)$ be the four clusters of the c-image C induced by two overlapping clusters $V(\mu)$ and $V(\nu)$ of W. In T, nodes α, β, and γ are the three children of λ, $G(\lambda)$ and $G(\gamma)$ are always connected, and at most one of $G(\alpha)$, $G(\beta)$ is not connected. Therefore C can be tested for c-planarity in $O(n^2)$ time.

In the hypothesis of Statement (b), C is a c-connected c-graph and therefore it can be tested for c-planarity by using [4, 5] or [2]. □

4 Final Remarks and Open Problems

This paper has introduced and studied a new problem in the field of cluster planarity, i.e., the overlapping cluster planarity testing problem. It has started the investigation by analyzing the not obvious relation with classical c-planarity testing; classes of oc-graphs have been described for which polynomial-time oc-planarity testing algorithms exist.

Several questions are naturally raised by the research described in this paper. Some of the most relevant in our opinion are listed below:

- It would be interesting to describe other meaningful families of oc-graphs for which the oc-planarity testing problem can be performed in polynomial-time. For example, what about oc-graphs containing three clusters at the same hierarchy level with a non empty intersection?

- It is well known that c-planar graphs can be tested for c-planarity in polynomial time if they are c-connected. What is the time complexity of testing a c-connected oc-graph for oc-planarity in the general case?

- A polynomial-time planarization algorithm is described in [6] for those c-graphs that are not c-planar. From the application side, it is important to investigate efficient planarization algorithms for oc-graphs that are not oc-planar or for which a polynomial-time planarity testing algorithm is not known.

- The approach used in this paper for studying the overlapping cluster planarity problem is to translate it to a classical c-planarity problem. Attacking the problem with a more direct approach is an interesting field of investigation.

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