Minimum Representations of Rectangle Visibility Graphs

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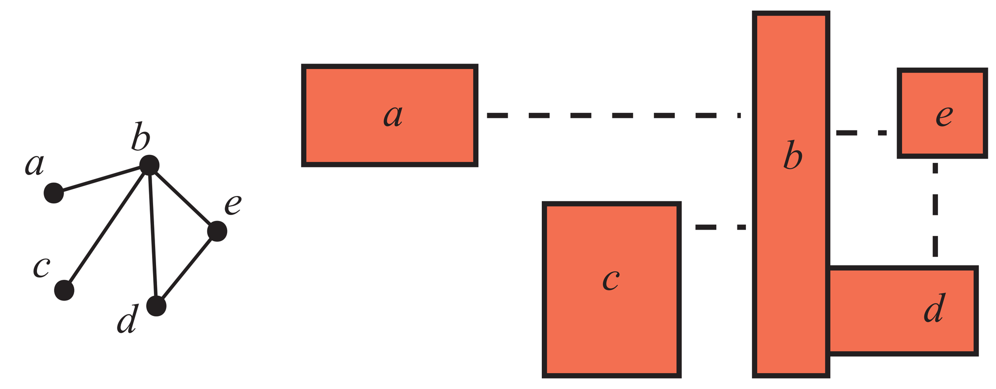
Definitions: Rectangle Visibility Graphs

Let R be a set of nonintersecting open rectangles in the plane with horizontal and vertical sides. Construct a graph G with a vertex for each rectangle in R, and an edge for each horizontal or vertical line of sight (zero-length lines of sight count).

G is a ***rectangle visibility graph*** (***RVG***) and R is its ***rectangle visibility representation***.



Main Question

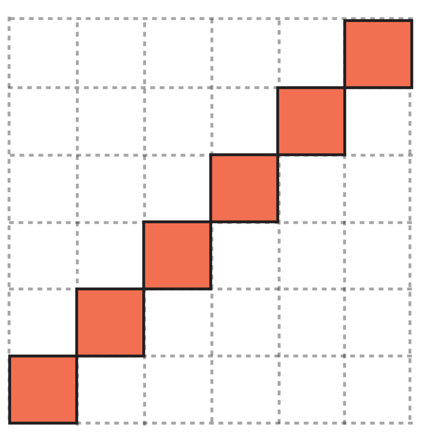
Suppose the corners of the rectangles are at integer coordinates. For a given RVG, how small can we make its representation?

Measures of Size

- area(G): the area of the smallest axis-parallel rectangle enclosing any representation of G.
- perimeter(G): the perimeter of this smallest enclosing rectangle.
- height(G): the length of the shorter side of this smallest enclosing rectangle.

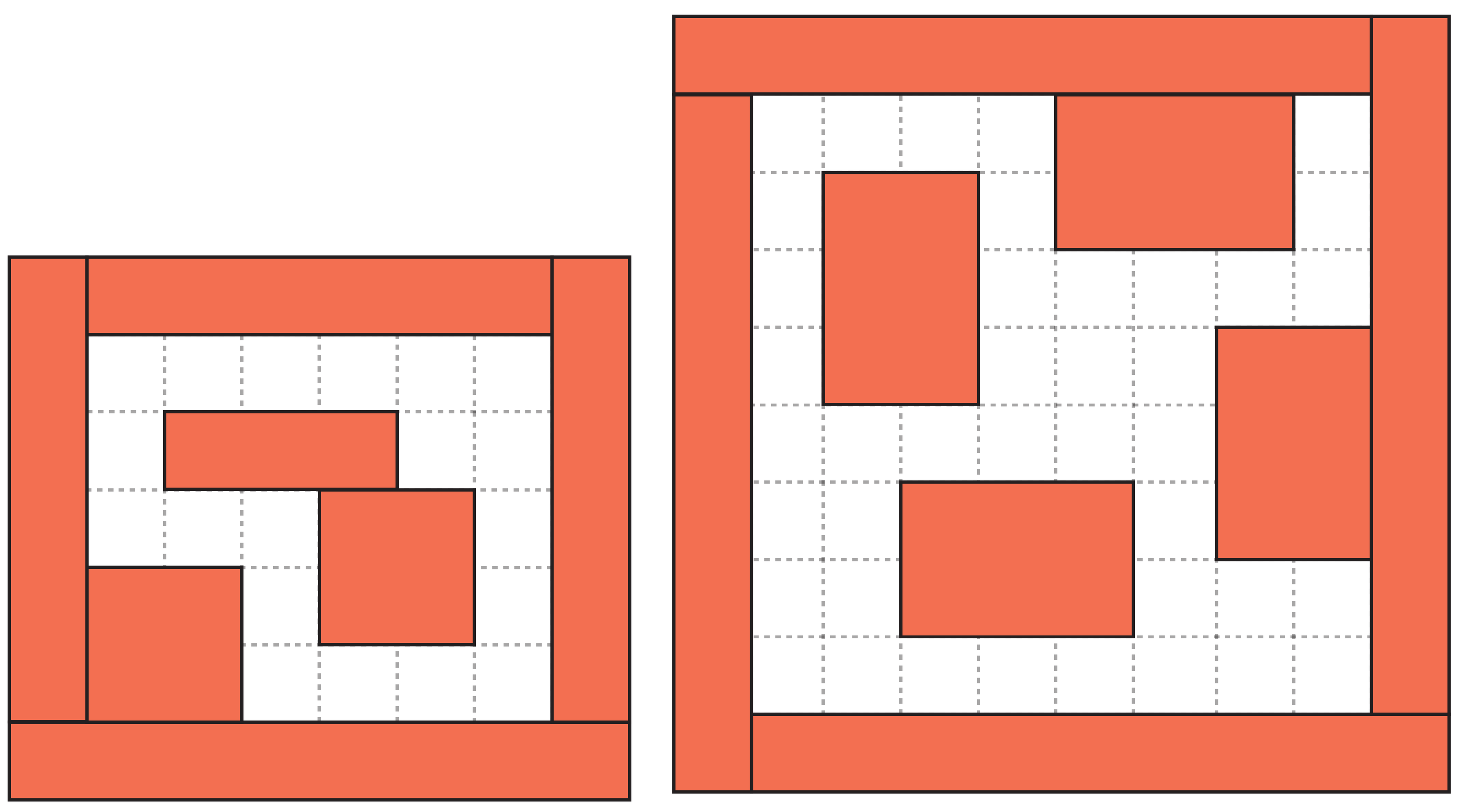
Areas of Graphs on n Vertices

Theorem. Among graphs with n vertices, $1 \le n \le 6$, the empty graphs E_n have largest area, n^2.



Theorem. Among graphs with n vertices, $n \ge 7$, the empty graphs E_n do not have largest area.

Proof. The graphs K_7 and K_8 must be enclosed by axis-parallel rectangles of dimensions at least 7×8 and 10×10, and so have area 56 and 100, respectively.



If $n \ge 9$ the disjoint union of K_8 and $n-8$ isolated vertices has area at least $(n-8+10)^2$, greater than the area of E_n.

Open Questions

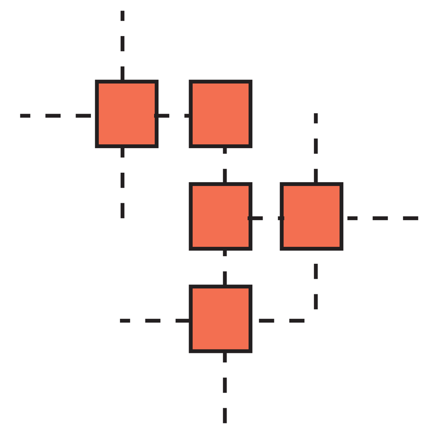
1. Are these the largest representations on 7 and 8 vertices? Since K_n is not an RVG for $n \ge 9$, which graphs need the largest representations on more than 9 vertices?
2. For any graph G with n vertices, $\mathrm{area}(G) \le 4n^2$. This bound is realized if all x- and y-coordinates of all rectangles are different. How close can we get to this bound?

Heights of Trees

Theorem. A tree with ℓ leaves has height $\lceil \ell/2 \rceil$ or $\lceil \ell/2 \rceil + 1$.

Step 1. A tree with ℓ leaves has height at least $\lceil \ell/2 \rceil$.

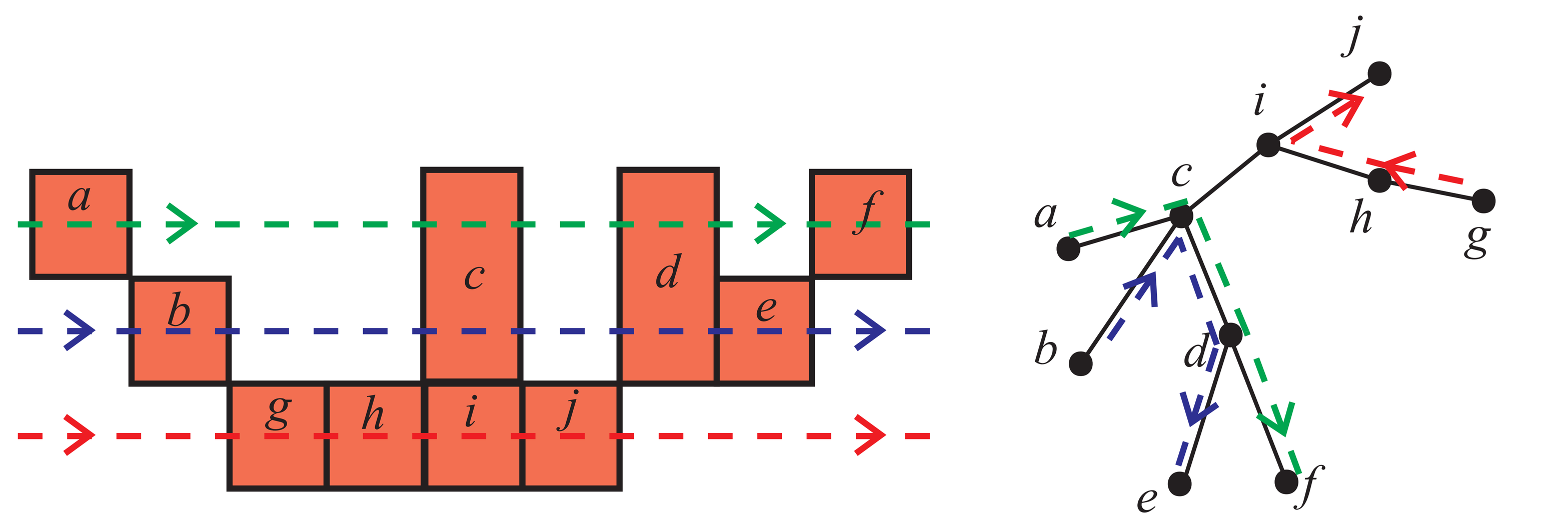
Proof. Each leaf-rectangle sees another rectangle on only one side, so each row and column in a representation has at most two leaves.



An ***orienting path cover*** of a tree T is a set of k directed paths in T containing all of its vertices, such that whenever two paths share an edge, they point the same direction.

Step 2. A tree has a representation with height k if and only if it has an orienting path cover with k paths.

Proof. Each path corresponds to a unit of height in the representation (the y-coordinates of the rectangles), and the common direction of the paths corresponds to the direction of increasing x-coordinates of the rectangles.
The height of each rectangle is the number of paths its vertex is in. Rectangles at different heights can be placed consistently if the directions of the paths agree.



Step 3. A tree with ℓ leaves has an orienting path cover with at most $\lceil \ell/2 \rceil + 1$ paths.

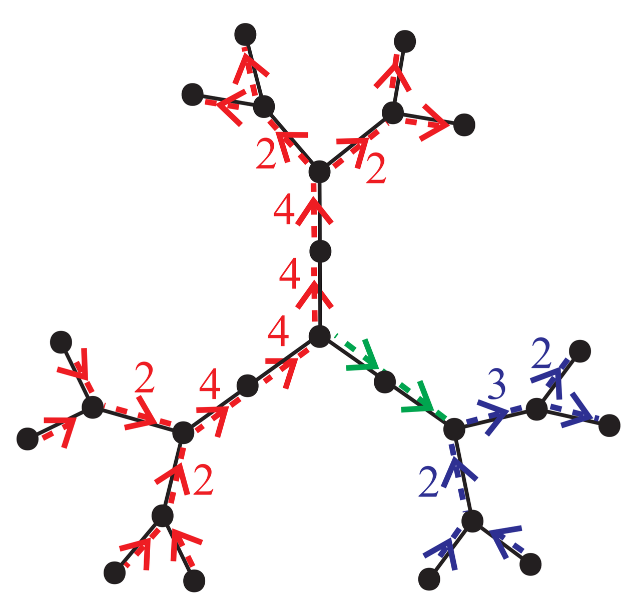
Proof. Note that every orienting path cover must include a path ending at every leaf. This is $\lceil \ell/2 \rceil$ paths. So equivalently, we show that T has an orienting path cover using at most one extra path.

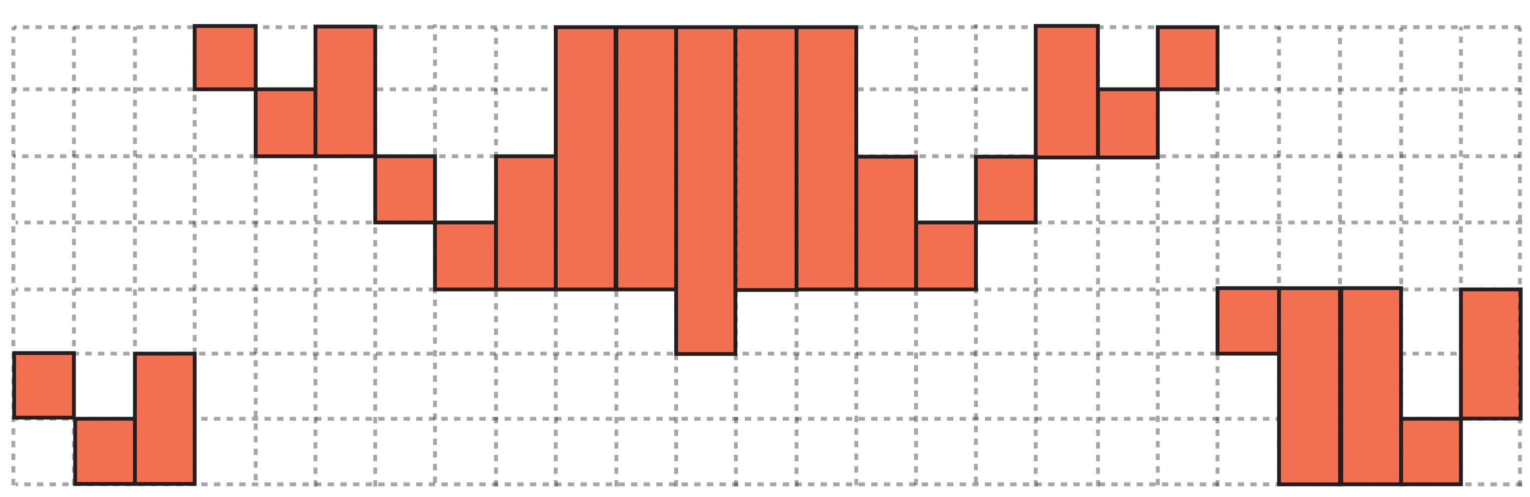
We construct the orienting path cover one vertex at a time. At each vertex v of the tree, we construct two edges of each path, one directed into v and one directed out.

Label each edge at v with the number of leaves on its branch. If an edge is labeled k, we want at most k paths on e. If we have exactly k paths on e, the paths can cover this branch, ending in every leaf.

Labeling at v Covering a branch

It is always possible to do this on all but one of the edges at v. The construction either continues along this edge, or in one family of special cases, starts the one extra path along this edge.





This tree has 12 leaves and height $7 = \lceil 12 \rceil + 1$, so this bound is tight.