Drawing Simultaneously Embedded Graphs with Few Bends

Luca Grilli Seokhee Hong Jan Kratochvíl Ignaz Rutter

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Simultaneous Embedding

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Task: Find sequence of drawings $\Gamma_1, \dots, \Gamma_k$ such that

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- straight-line (SIMULTANEOUS GEOMETRIC EMBEDDING)
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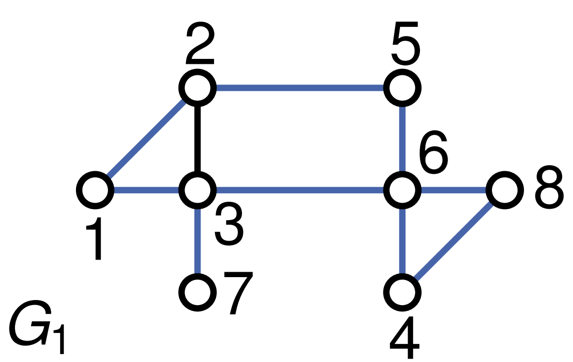
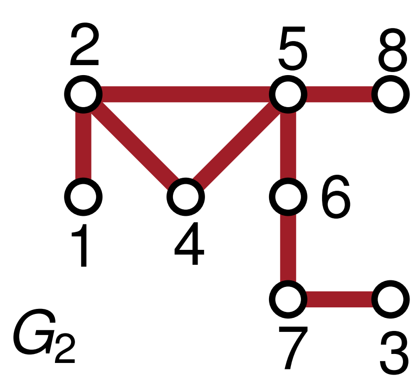
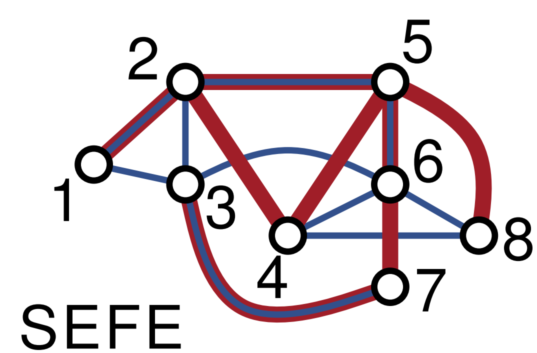
Restrict to $k = 2$.







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No testing algorithm for any non-trivial class of graphs!

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Complexity for two graphs open!

But: Several algorithms for restricted cases.

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G_1 and G_2 with common graph G admit SEFE

$\Leftrightarrow$

there exist planar embeddings $\mathcal{E}_1$, $\mathcal{E}_2$ of G_1, G_2 with $\mathcal{E}_1|_G = \mathcal{E}_2|_G$.

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Given a SEFE embedding, how do we get a good SEFE drawing?

- In particular: How many bends per edge for polyline drawings?

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Can test existence/find SEFE Embedding for a lot of graphs.

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If common graph has no edges, three bends per edge suffice.

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Draw one graph straight line, the other $O(|V(G)|)$ bends.

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(c_1, c_2)-drawing: common edges $\leq c_1$ bends, exclusive $\leq c_2$ bends.
What value pairs are possible for every SEFE Embedding?

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G induced subgraph of G_1, G_2?

connectivity of G

	induced	general
2-connected		
connected		
components are 2-connected		
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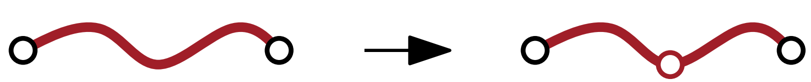

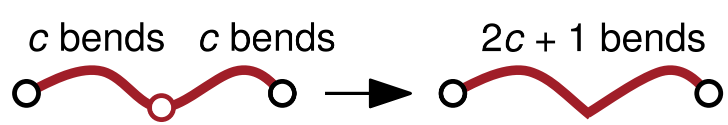

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- SEFE Embedding of k graphs with sunflower intersection may require $c \in \Omega(\sqrt{2}^k / k)$ for a $(0, c)$-drawing

Outline

1. SEFE Drawings when the common graph is (bi-)connected
2. The general case
3. Lower bounds

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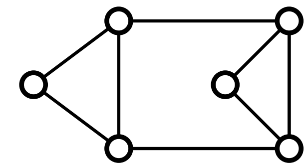
plane graph $\equiv$ planar graph with fixed embedding  G

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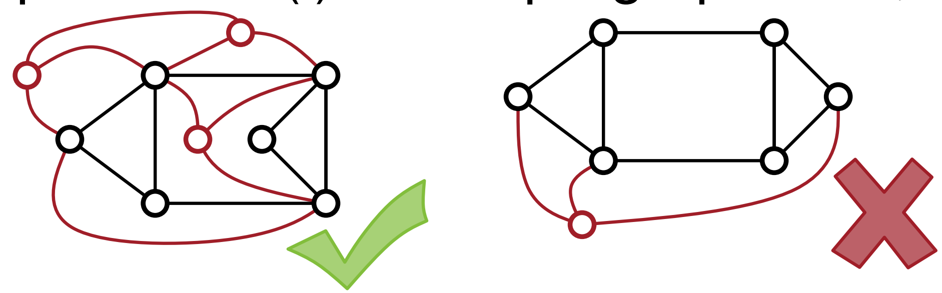

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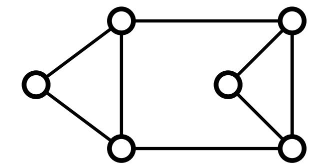

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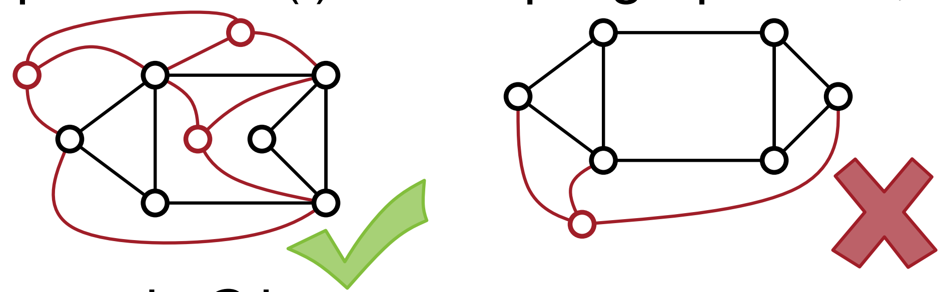

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k-universal: can be extended to *k*-bend drawing of every plane supergraph

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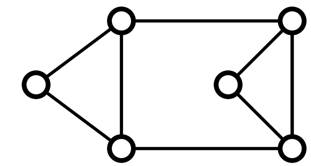
plane graph $\equiv$ planar graph with fixed embedding  G

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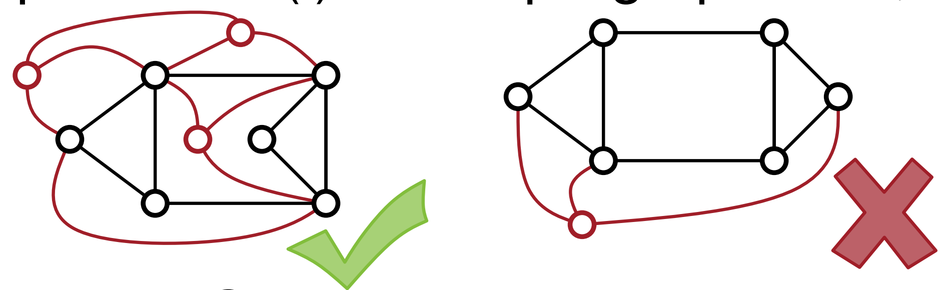

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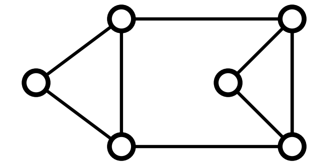

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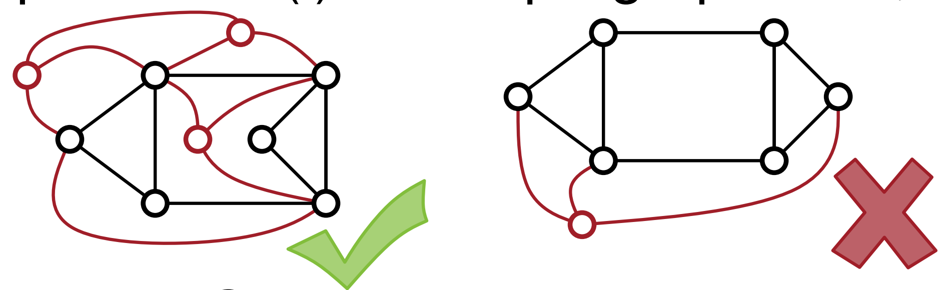

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1. Take (induced) k-universal drawing of common graph G
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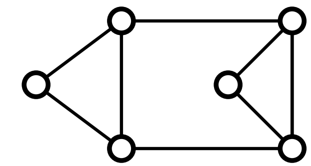
plane graph $\equiv$ planar graph with fixed embedding  G

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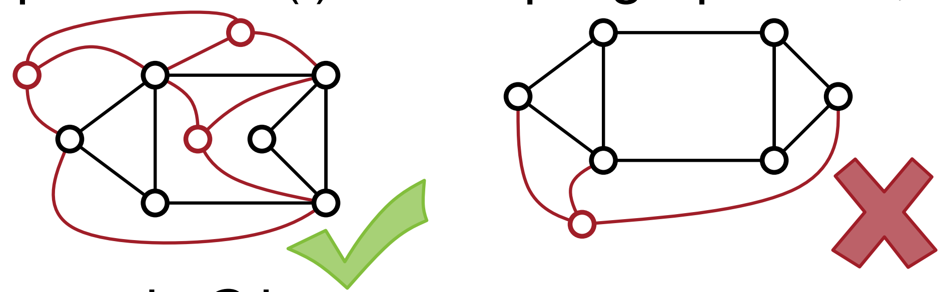

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Do they even exist?!

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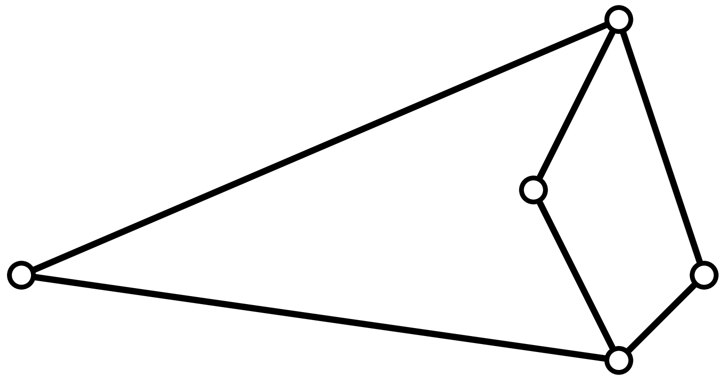
Every biconnected planar graph has an induced 0-universal drawing.

Proof:
- take star-shaped drawing



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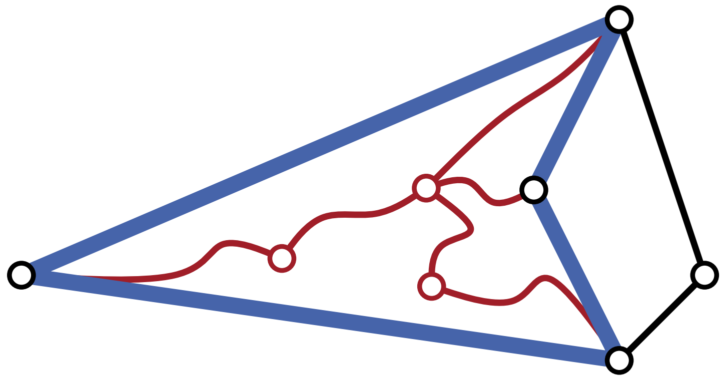

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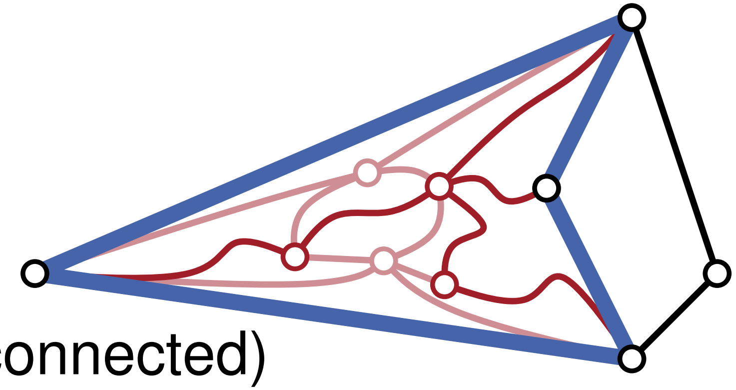
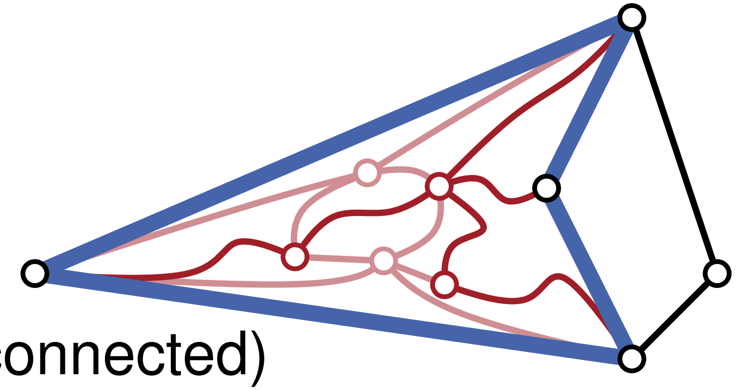


Universal Drawings of Biconnected Graphs

Theorem
Every biconnected planar graph has an induced 0-universal drawing.

Proof:
- take star-shaped drawing



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Internally triconnected planar graph, outer face fixed to starshaped-polygon
⇒ can be extended without bends [Hong, Nagamochi '08] □

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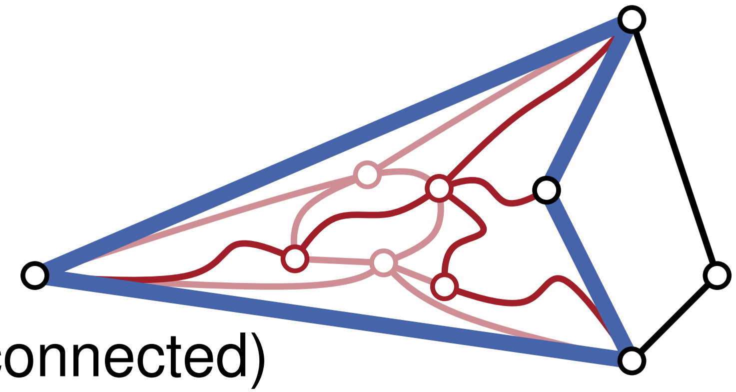

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Internally triconnected planar graph, outer face fixed to starshaped-polygon
$\Rightarrow$ can be extended without bends [Hong, Nagamochi '08] $\square$

Corollary

SEFE Embedding with G biconnected + induced admits $(0, 0)$-drawing.

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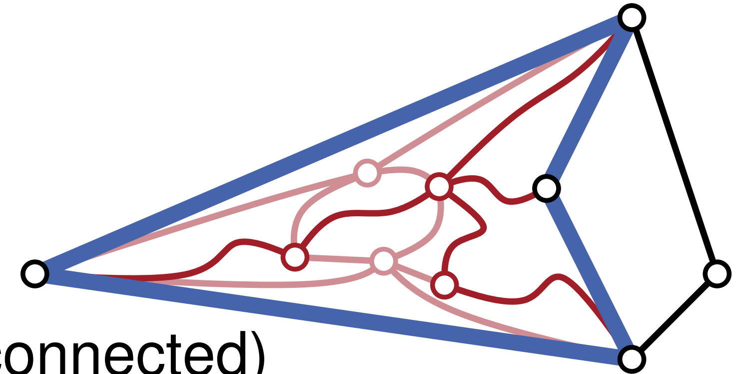

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Corollary

Every biconnected planar graph has a 1-universal drawing.
SEFE Embedding with G biconnected admits $(0, 1)$-drawing.

G induced subgraph of G_1, G_2?

connectivity of G

	induced	general
2-connected	(0, 0)-drawing	(0, 1)-drawing
connected	(0, 1)-drawing	(0, 3)-drawing
components are 2-connected	(0, 3)-drawing	(0, 7)-drawing
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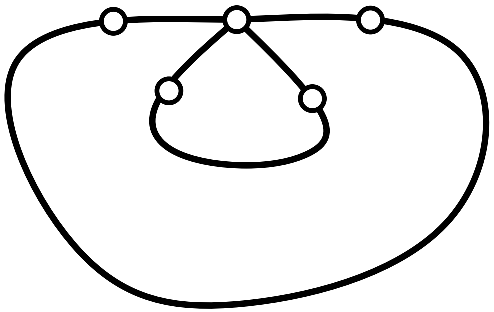


Universal Drawings of Connected Graphs

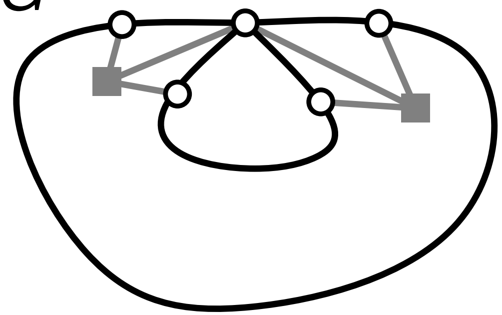
Theorem

Every connected planar graph has an induced 1-universal drawing.

Proof: add angle vertices at all angles of cutvertices of G

resulting graph G' is biconnected

$\Rightarrow$ has induced 0-universal drawing



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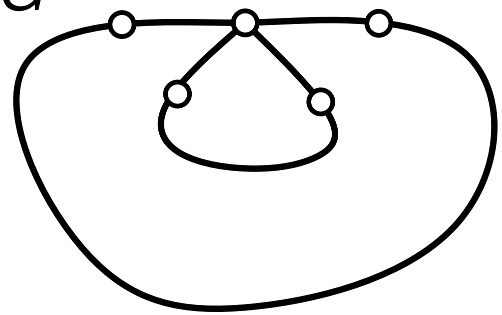

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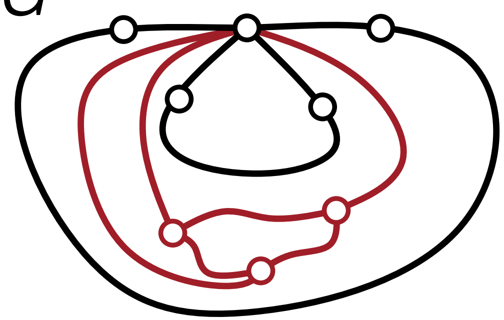

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Consider plane supergraph H of G.

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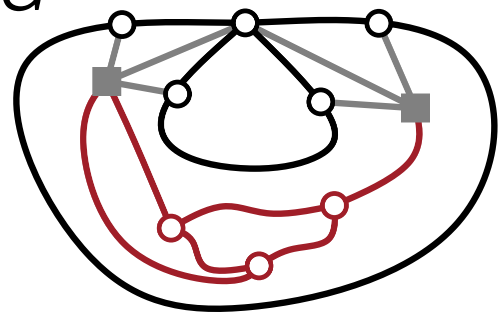

Consider plane supergraph H of G.

Consider G', at cutvertices reconnect H-edges to angle vertices.

$\Rightarrow$ resulting graph H' is plane supergraph of G'

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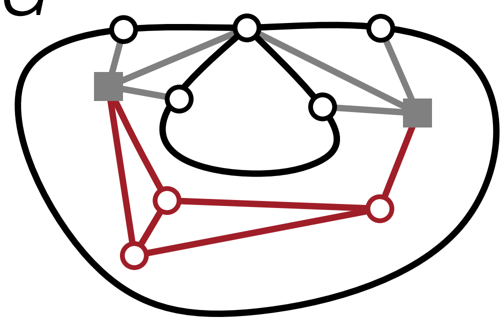

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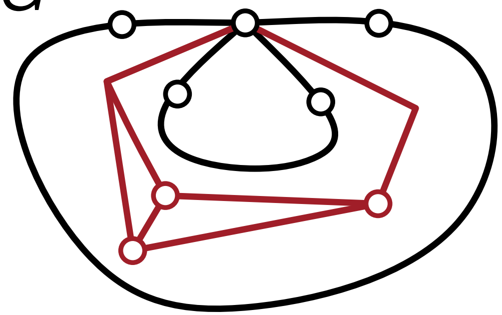

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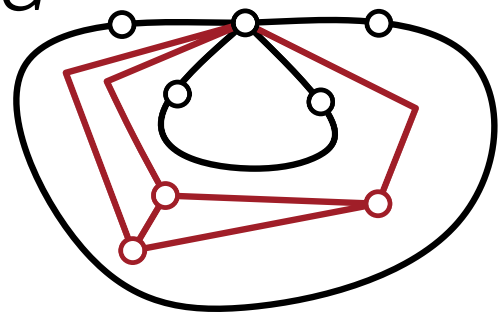

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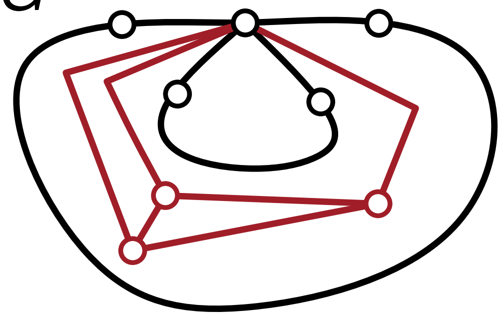

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Corollary

SEFE Embedding with G conntected + induced admits $(0, 1)$-drawing.

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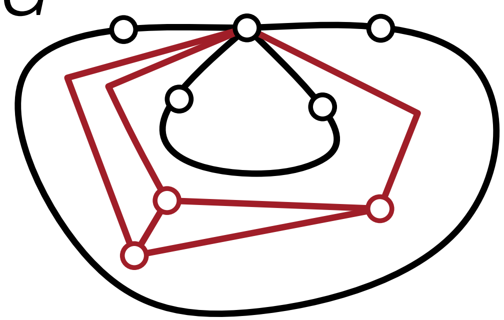

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Corollary

Every connected planar graph has a 3-universal drawing.

SEFE Embedding with G connected admits $(0, 3)$-drawing.

G induced subgraph of G_1, G_2?

connectivity of G

	induced	general
2-connected	(0, 0)-drawing	(0, 1)-drawing
connected	(0, 1)-drawing	(0, 3)-drawing
components are 2-connected	(0, 3)-drawing	(0, 7)-drawing
arbitrary	(0, 4)-drawing	(0, 9)-drawing

Universal Drawings of General Graphs

There are disconnected graphs that do not admit a $O(1)$-universal drawing.

In fact: (induced) k-universal drawings of G requires $k \in \Omega(|V(G)|)$

[Gordon '14]

Previous talk: Every drawing of G is (induced) $O(|V(G)|)$-universal.

[Chan, Frati, Gutwenger, Lubiw, Mutzel, Schaefer]

Need another approach to handle general graphs.

Outline

1. SEFE Drawings when the common graph is (bi-)connected
2. The general case
3. Lower bounds

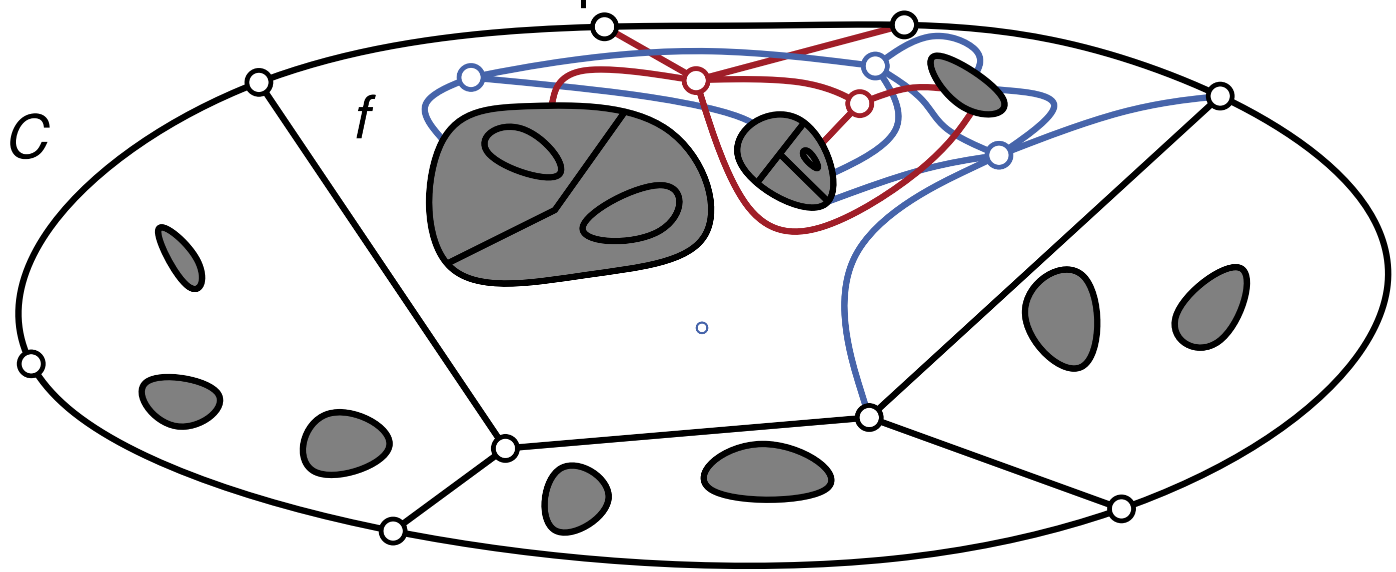
Connected Components are Biconnected

Theorem

SEFE Embedding with G induced + every component of G biconnected admits $(0, 3)$-drawing.

Draw from inside out. Assume:

- inner parts with all exclusive edges inside already drawn
- outer face star-shaped



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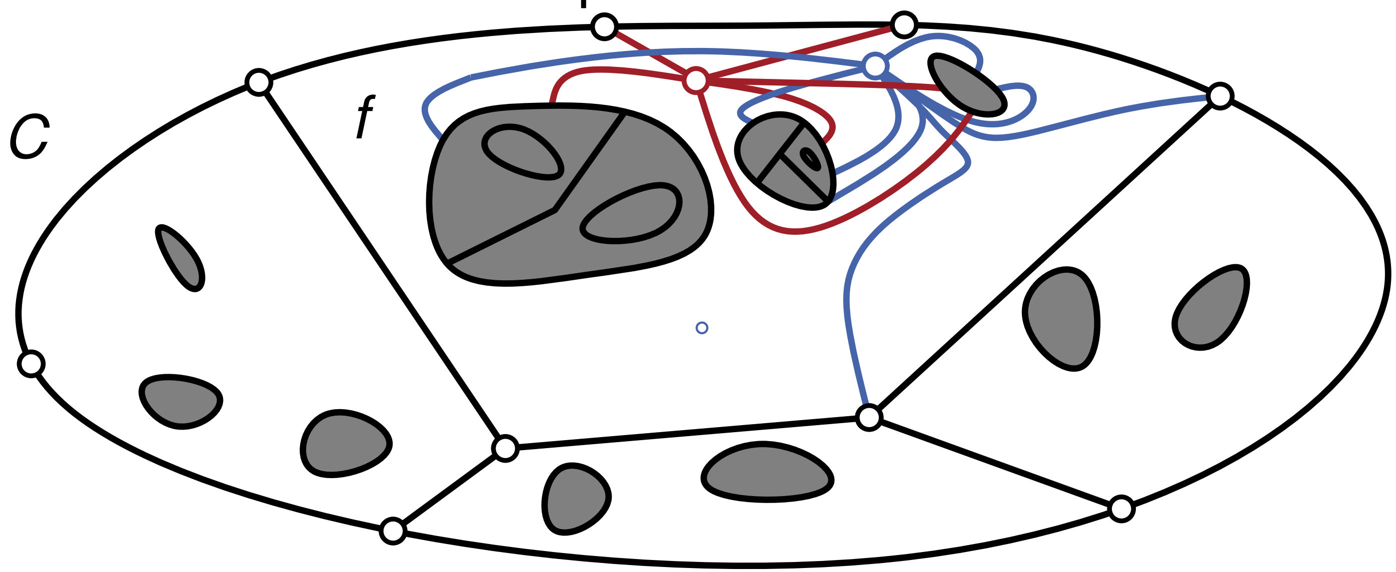

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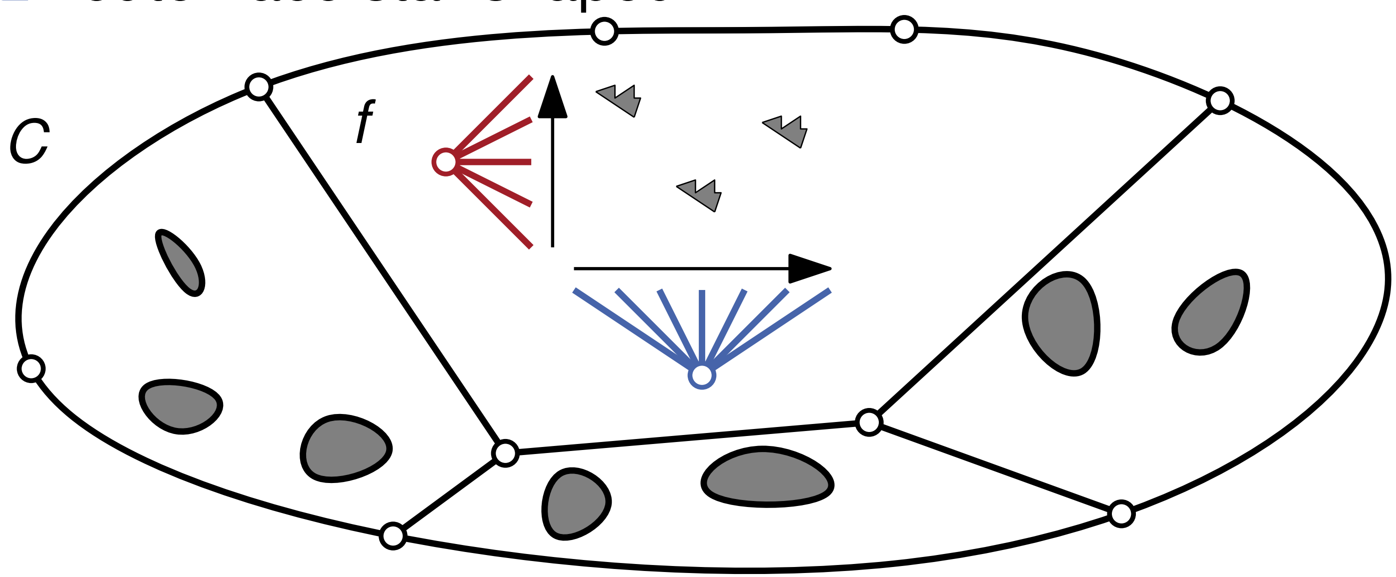

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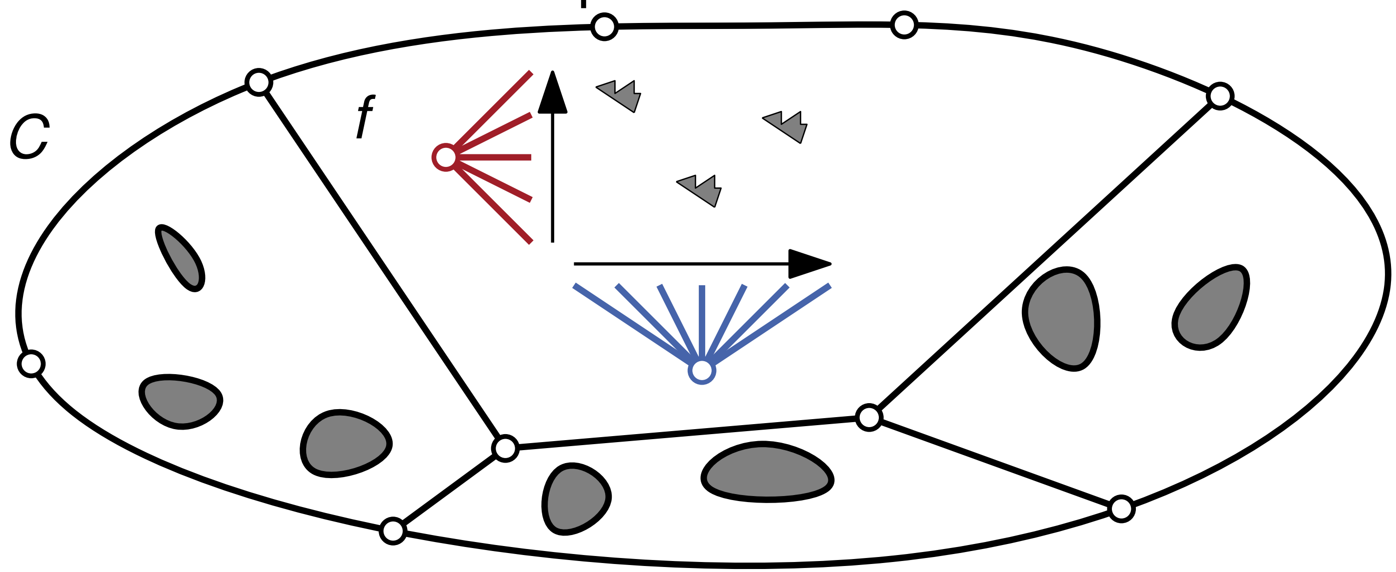

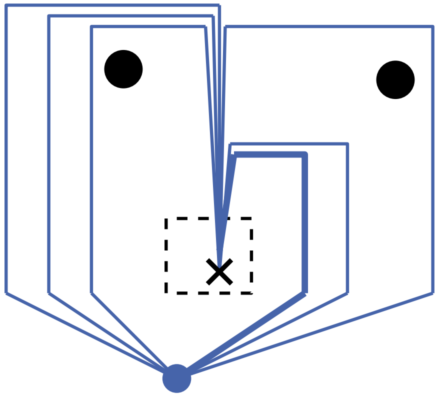

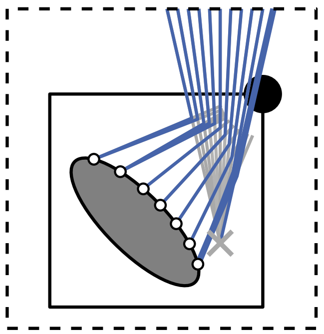

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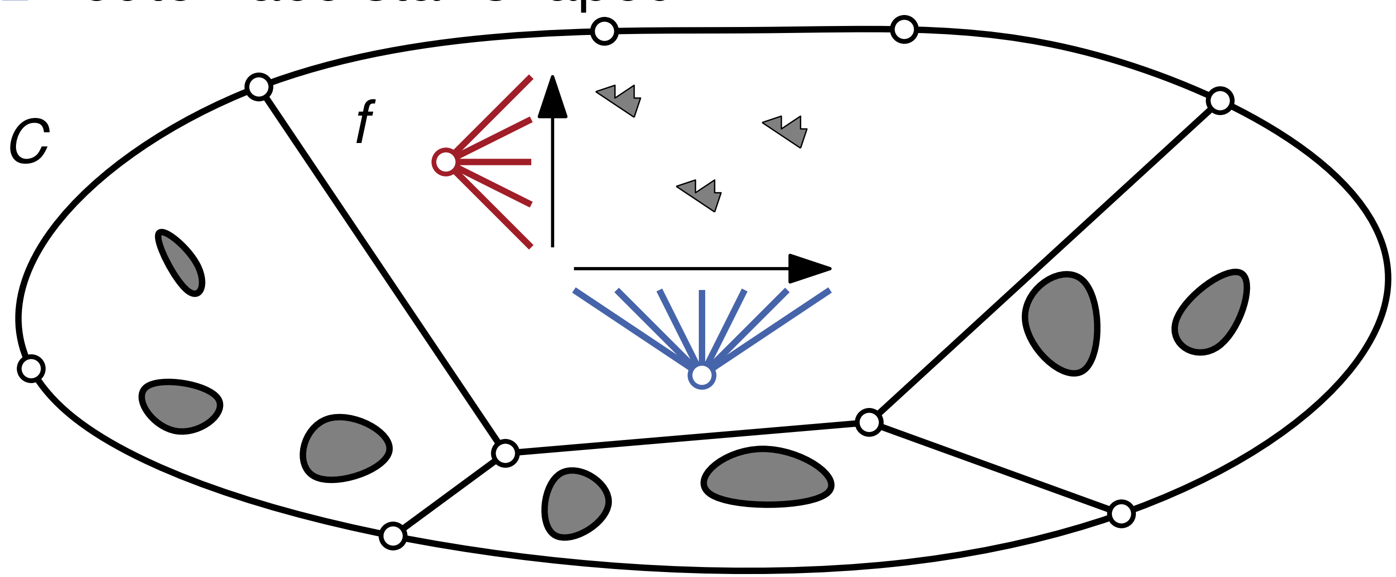

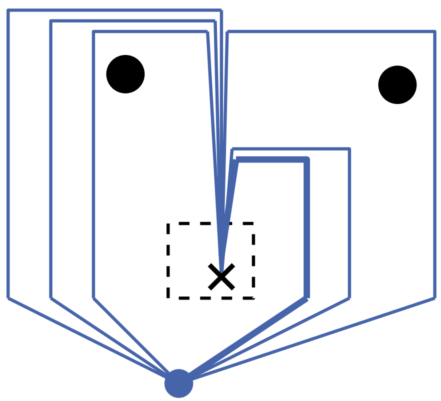

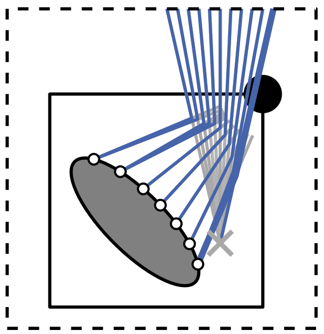

Save first bend on each edge, undo contraction.

□

Remaining Cases

Theorem

SEFE Embedding with G induced + every component of G biconnected admits $(0, 3)$-drawing.

Treat cutvertices as before:
$\Rightarrow$ $(0, 4)$-drawing for common graph induced

Subdivide exclusive edges to make G induced:
$\Rightarrow$ $(0, 9)$-drawing

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2. The general case
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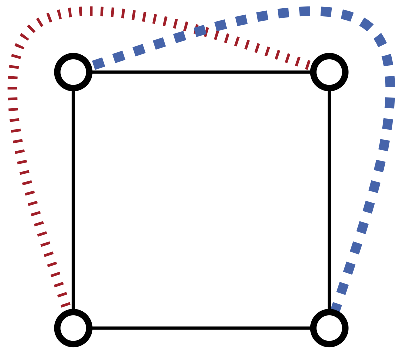
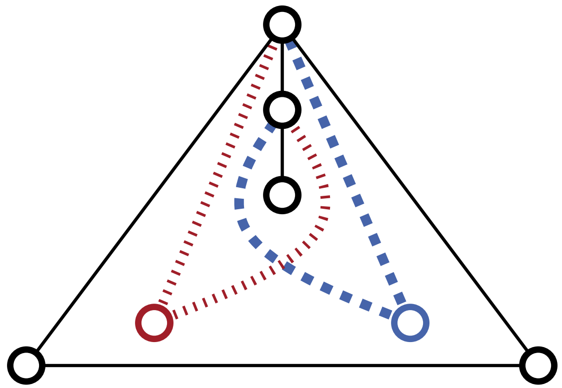
Lower Bounds

Instances requiring at least one bend on an exclusive edge:





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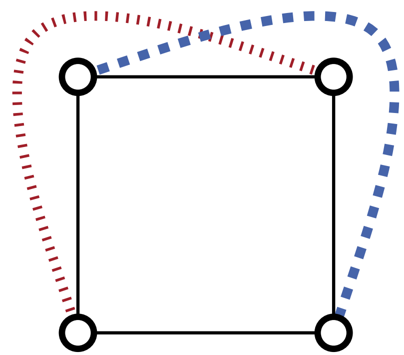

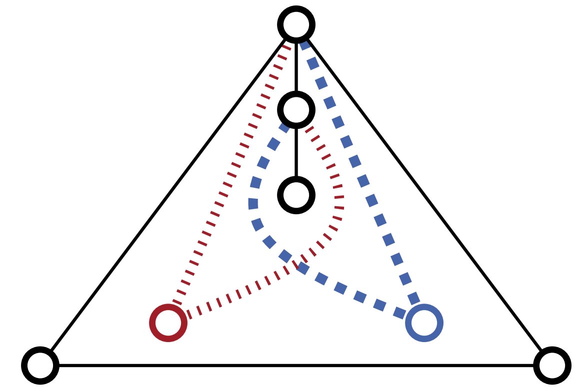

Theorem

There is a family of SEFE Embeddings of k graphs with sunflower intersection such that any $(0, c)$-drawing has $c \in \Omega(\sqrt{2}^k / k)$.

Proof:

- based on lower bounds on crossing number
- edges that cross often need to bend often

Our Results

G induced subgraph of G_1, G_2?

connectivity of G

	induced	general
2-connected	$(0, 0)$-drawing	$(0, 1)$-drawing
connected	$(0, 1)$-drawing	$(0, 3)$-drawing
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- $O(n)$-Algorithm for SGE when common graph is biconnected, induced.
- SEFE Embedding of k graphs with sunflower intersection may require $c \in \Omega(\sqrt{2}^k / k)$ for a $(0, c)$-drawing

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Open questions:

- Better upper/lower bounds.
- Better trade-off: bends on exclusive edge $\leftrightarrow$ bends on common edges?
- $(0, c)$-drawing for three graphs with sunflower intersection?