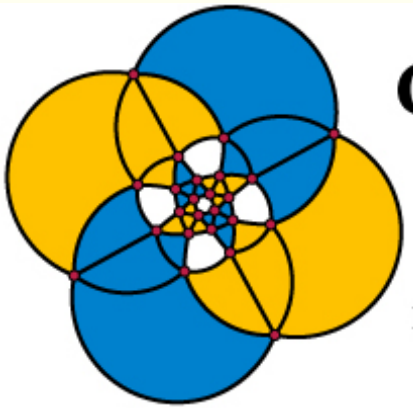
The Galois Complexity of Graph Drawing

Michael J. Bannister

William E. Devanny

David Eppstein

Michael Goodrich



Center for Algorithms and Theory of Computation

Donald Bren School of Information and Computer Sciences
University of California, Irvine

Overview

Motivation

Galois theory

Models of computation

Results

- Undrawable graphs!

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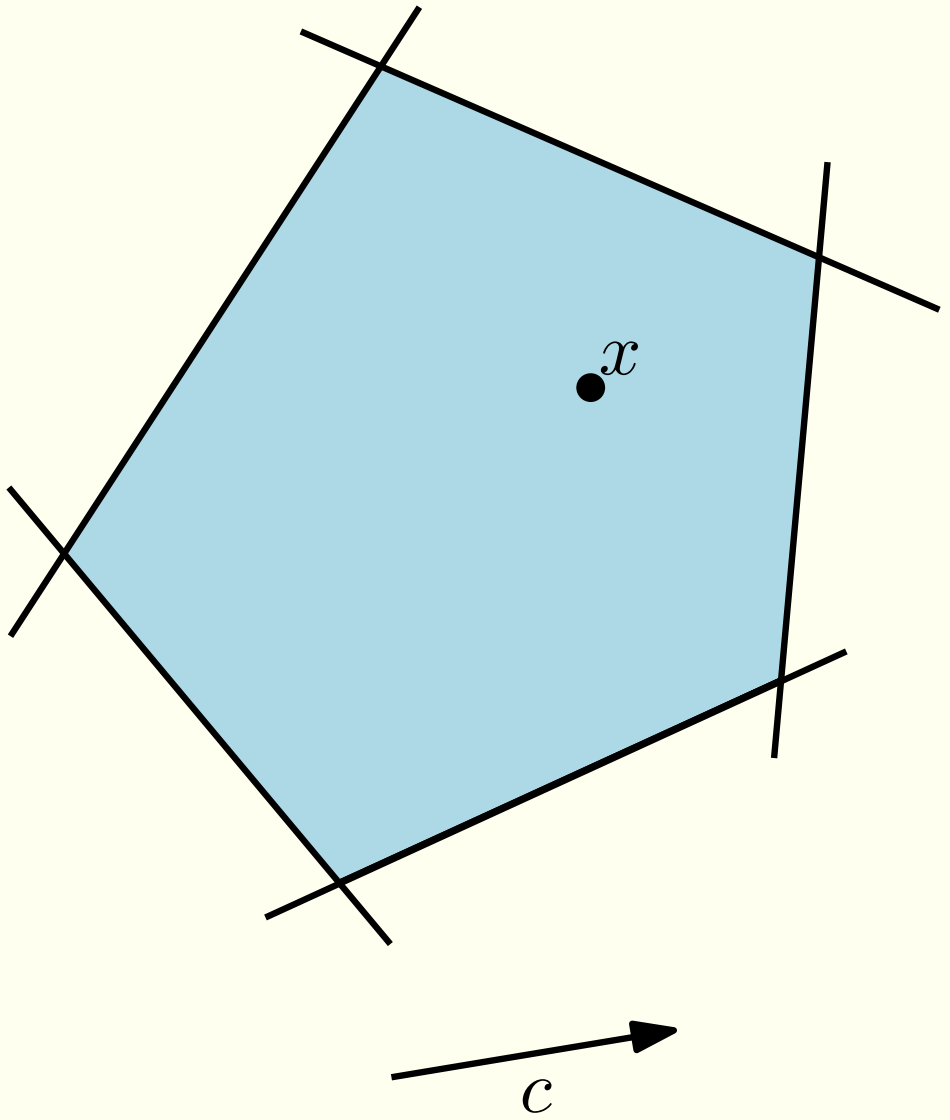
Models of computation

Results

- Undrawable graphs!

For some definition of undrawable

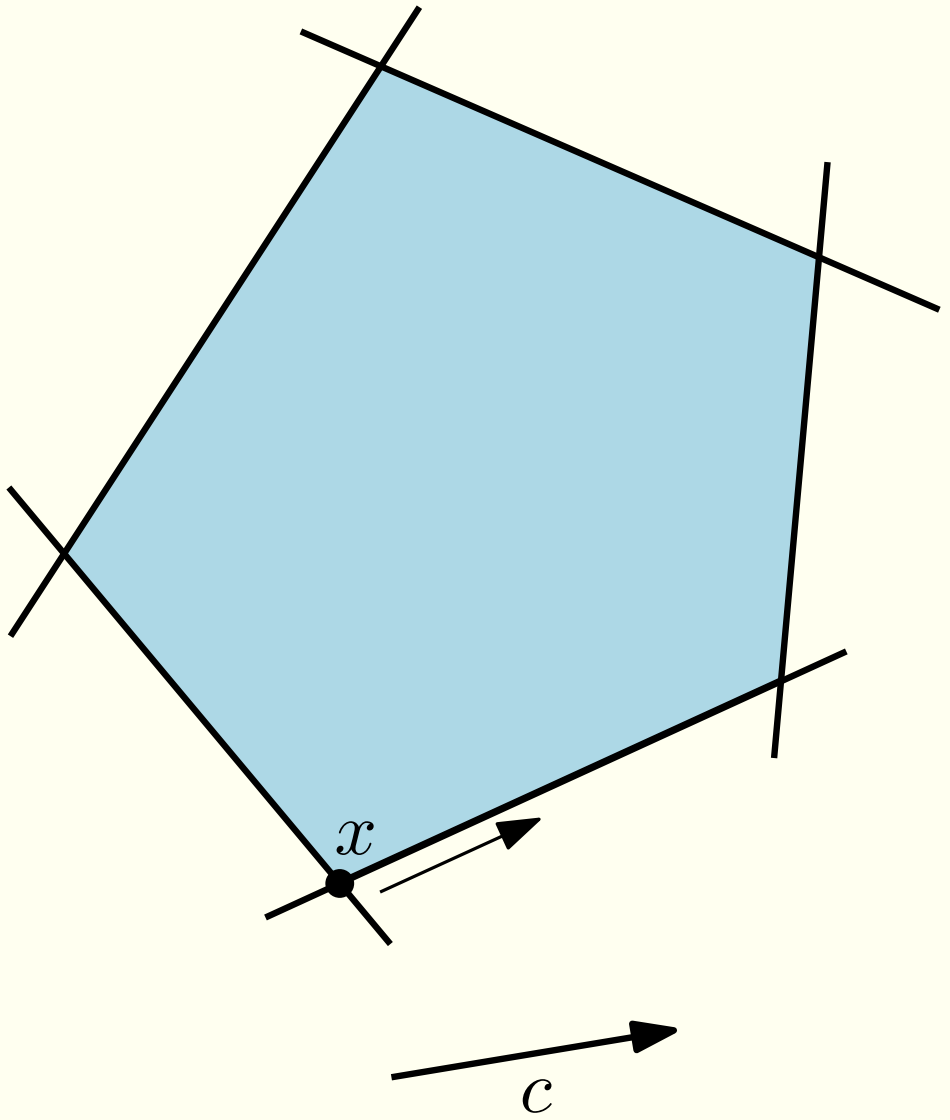
Motivation



$\max c^T x$

Simplex vs Interior Point

Motivation



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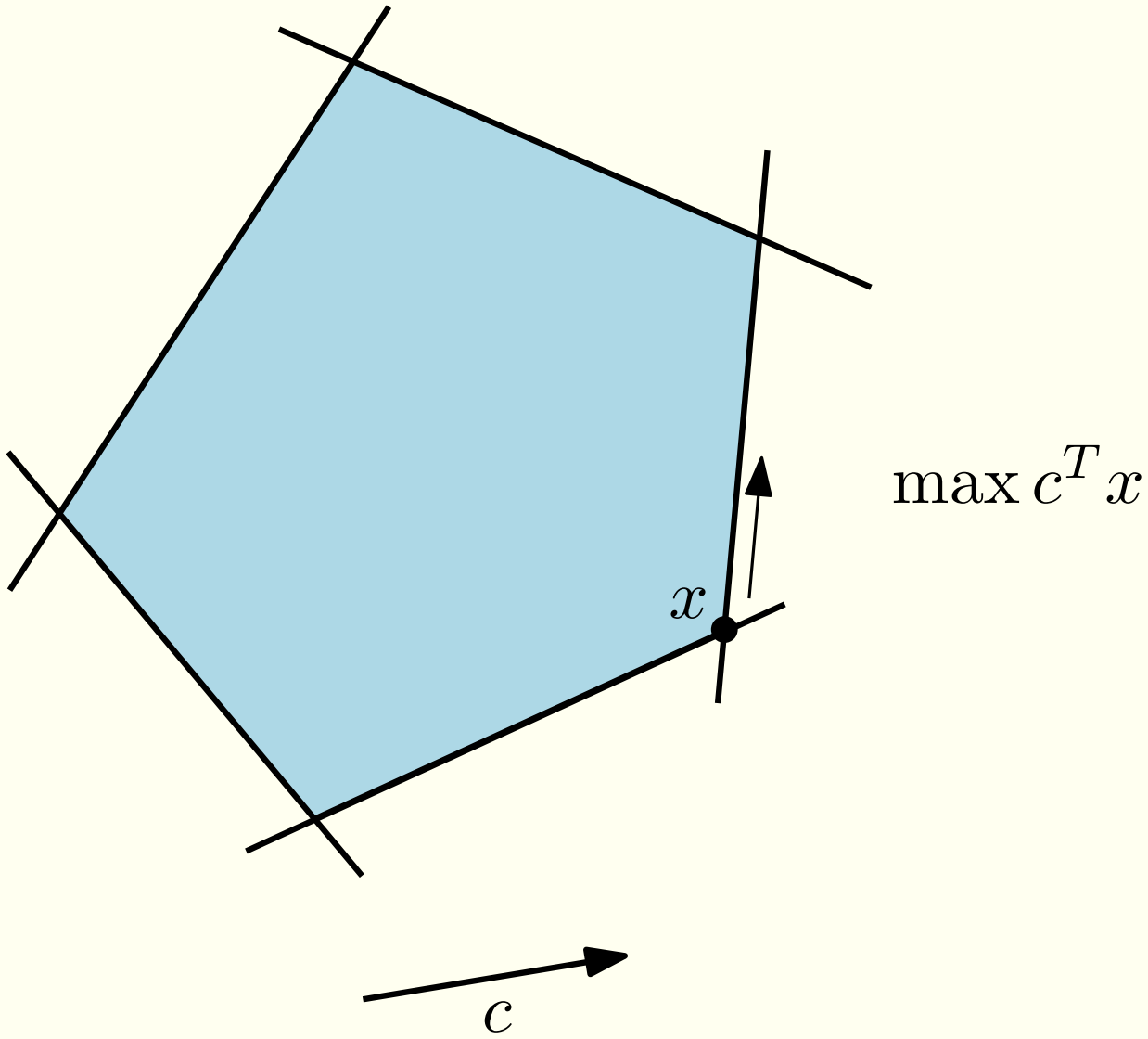

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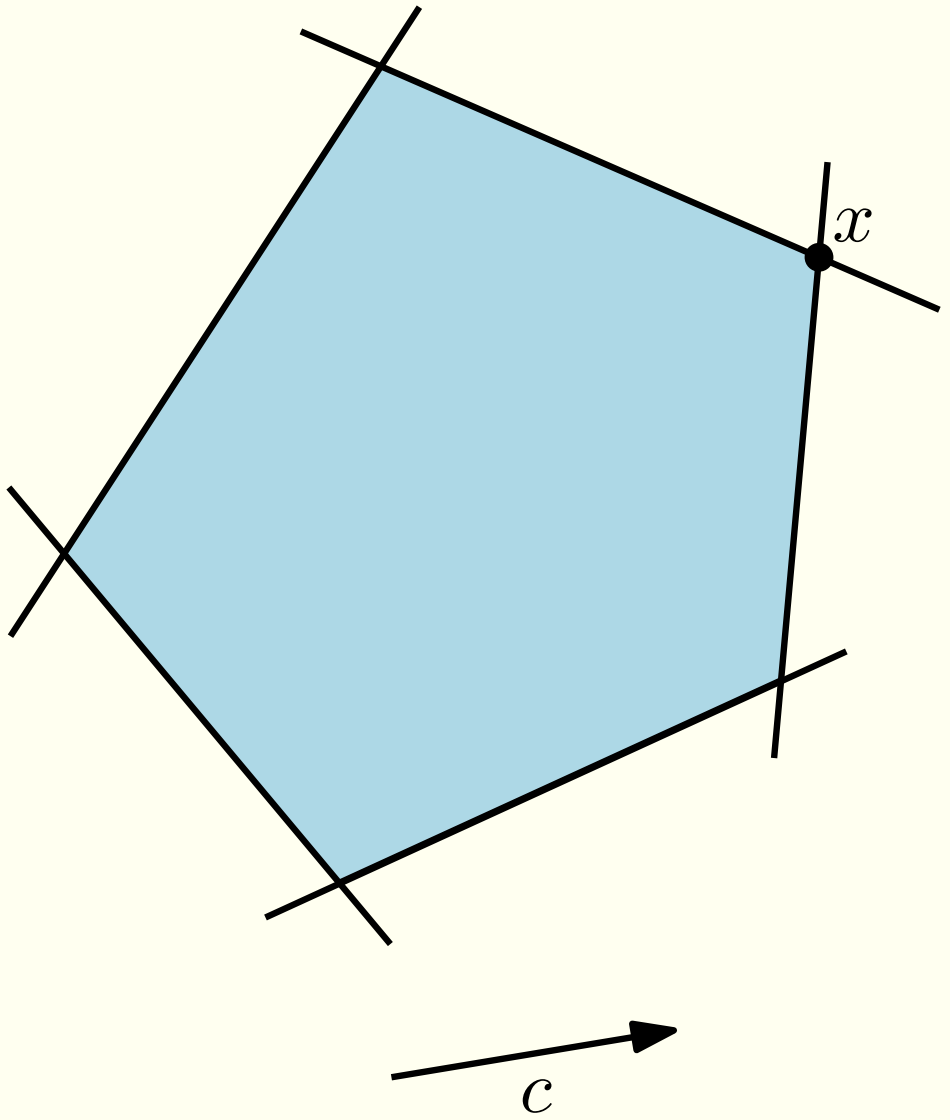

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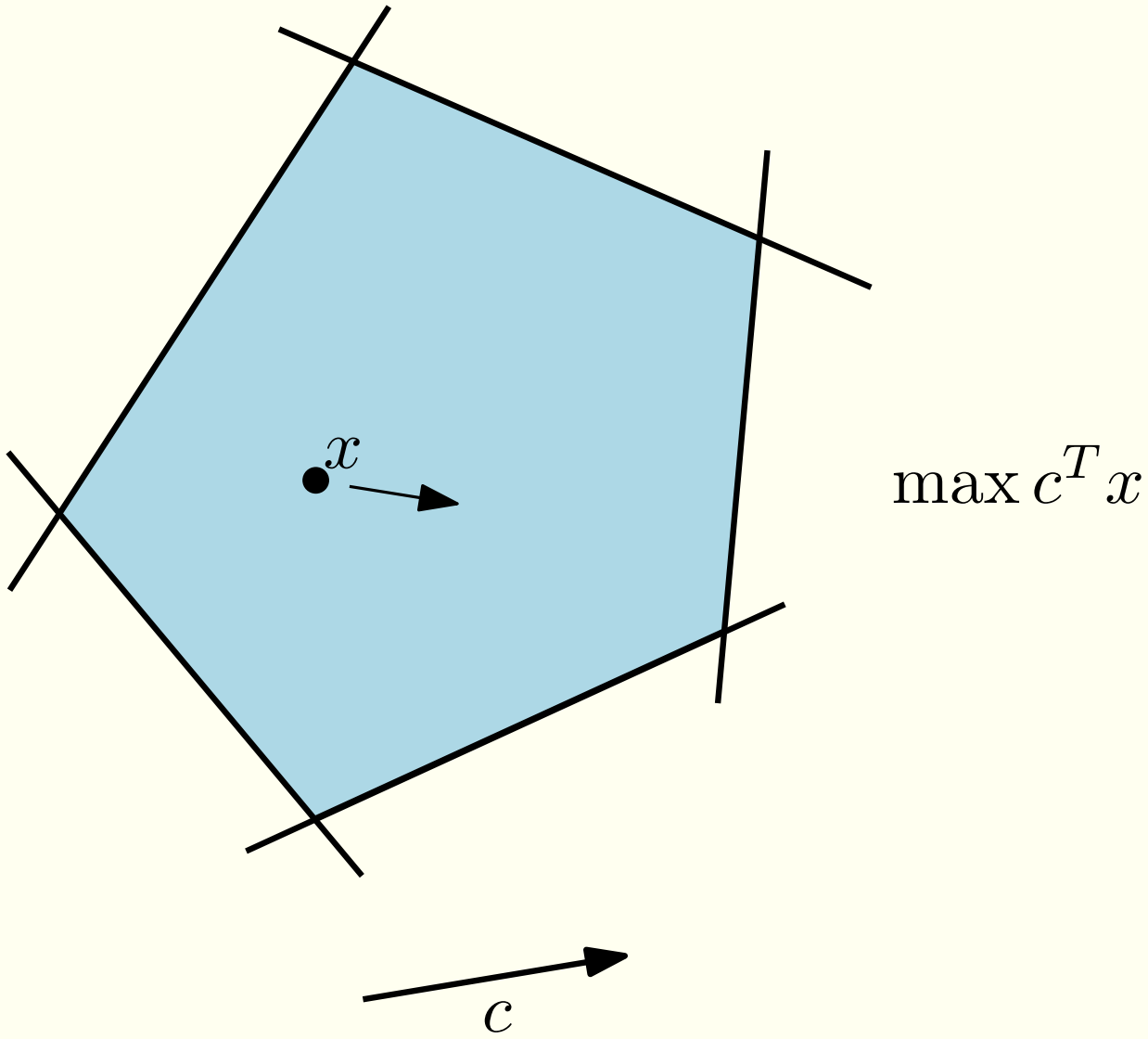

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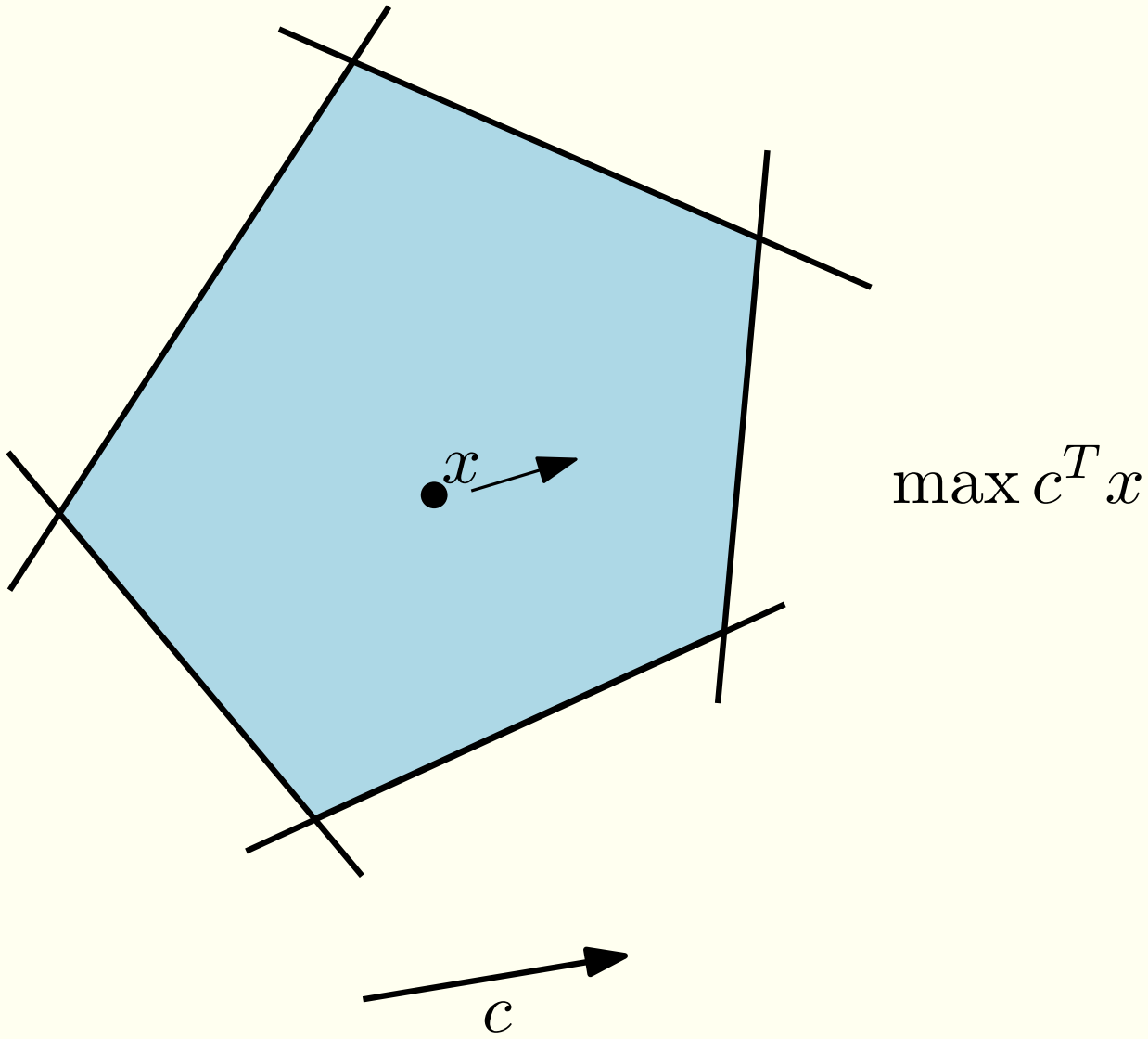

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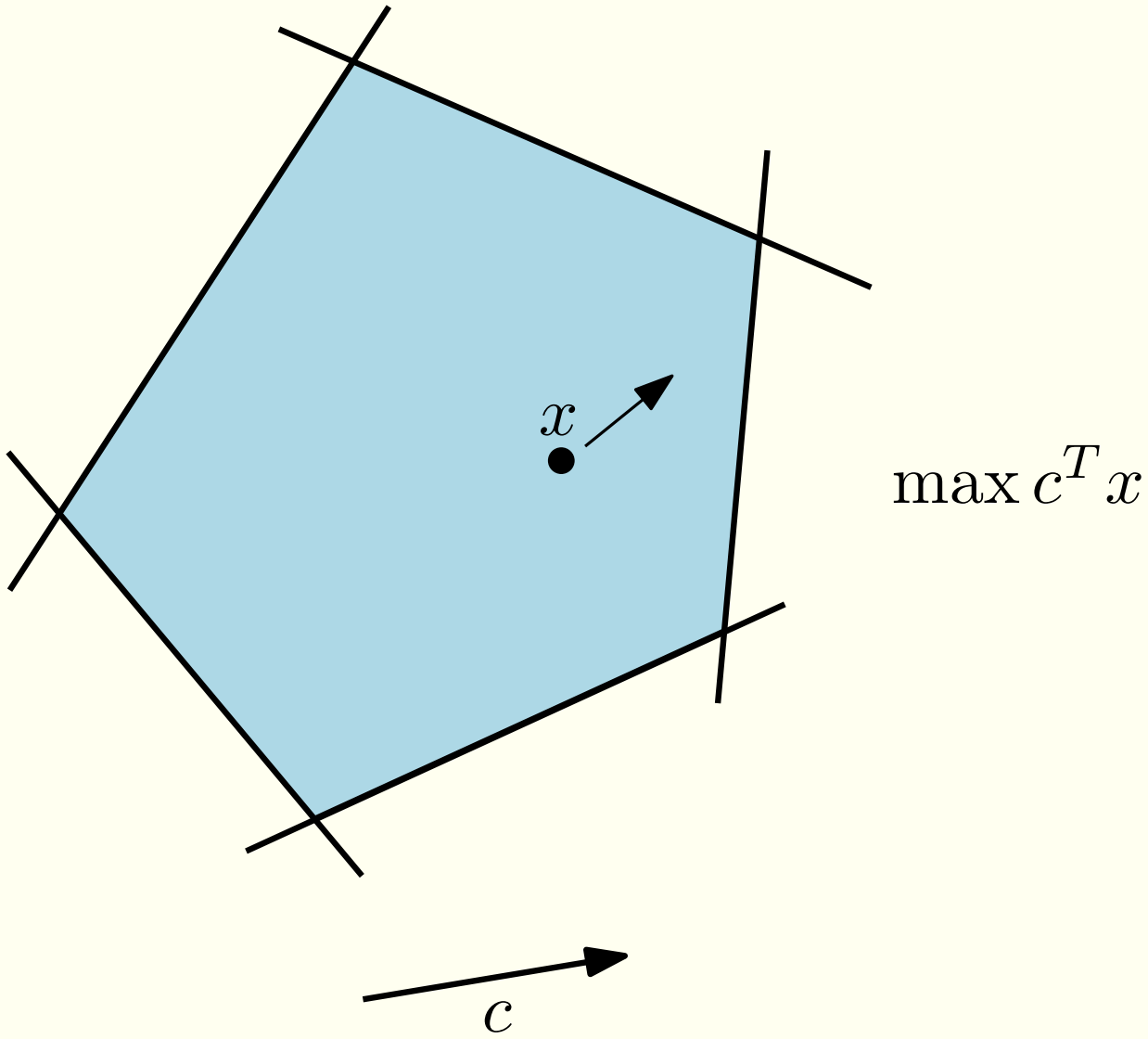

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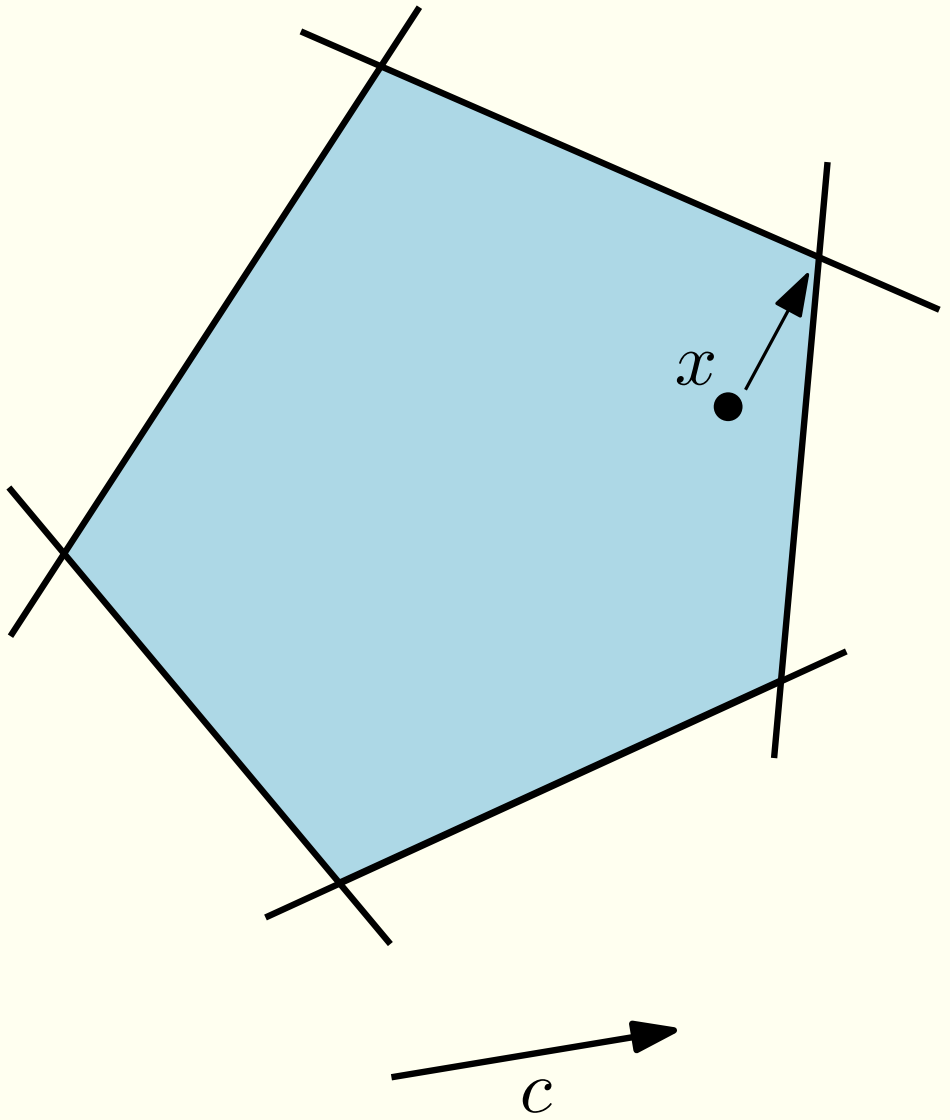

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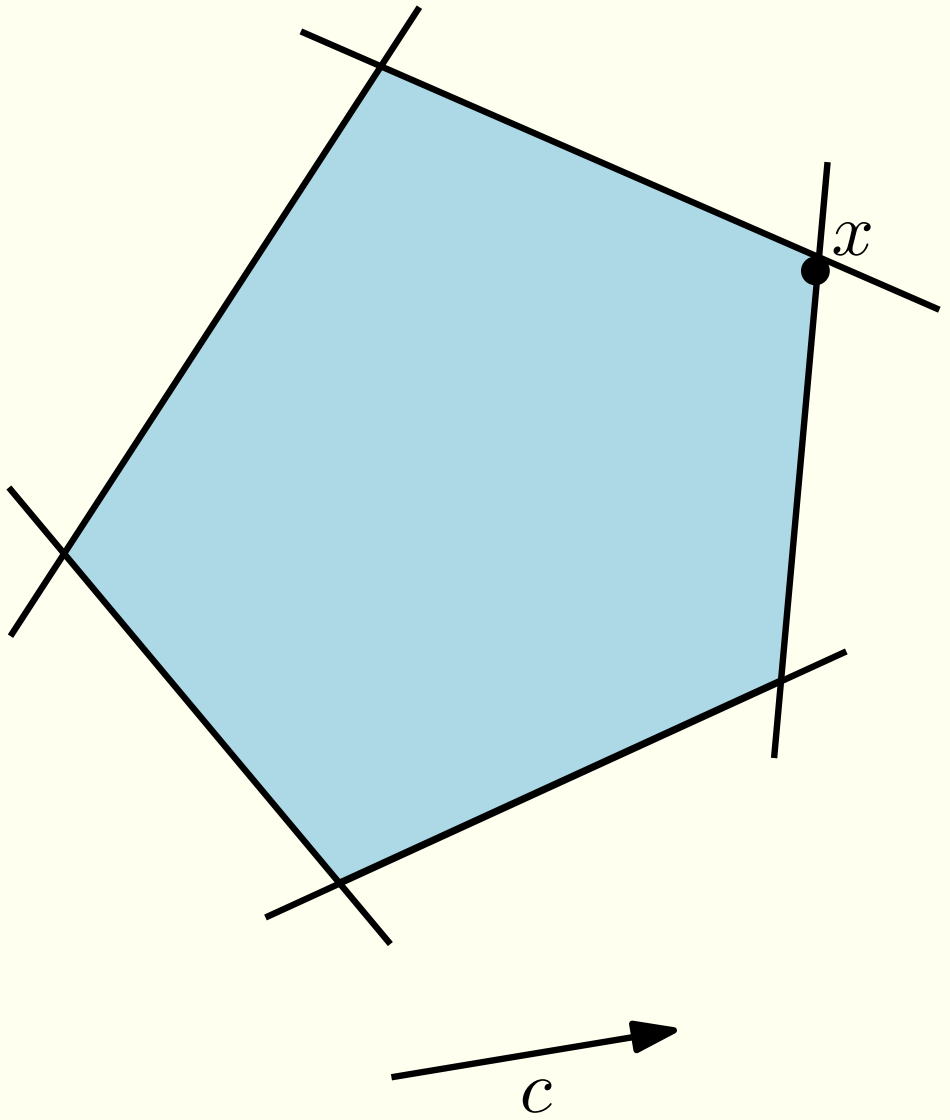

$$\max c^T x$$

Motivation

Symbolic vs. Numerical algorithms

Symbolic - manipulate mathematical expressions to obtain an exact answer for a problem

Simplex method

Numerical - iteratively walk towards the answer, improving an approximate answer with each step

Interior point method

Force Directed Graph Drawing

(Fruchterman and Reingold)

Neighbors:

$$f_a(d) = d^2/k$$

All pairs:

$$f_r(d) = k^2/d$$

When the total force at each vertex is zero, we are at F. and R. equilbrium

Also disqualify unstable/degenerate solutions

Force Directed Graph Drawing

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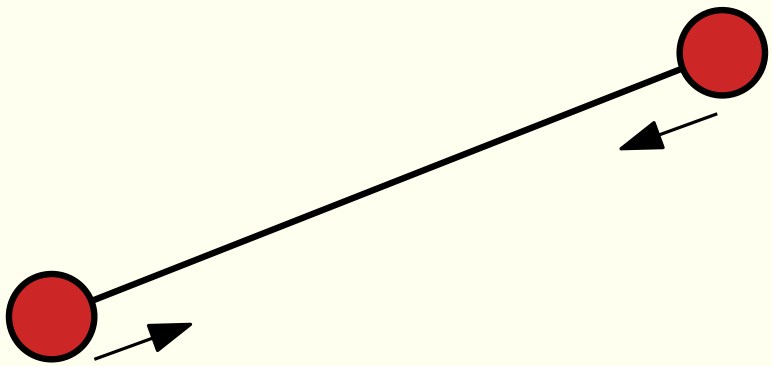

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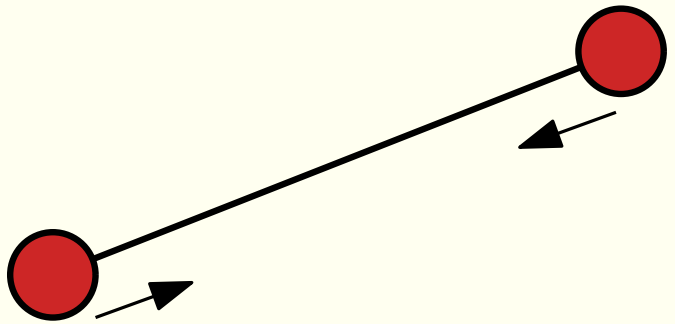

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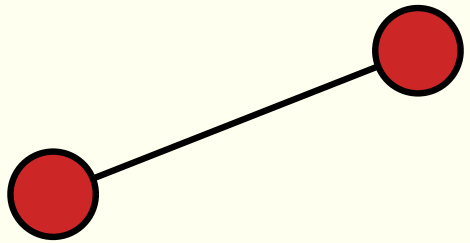

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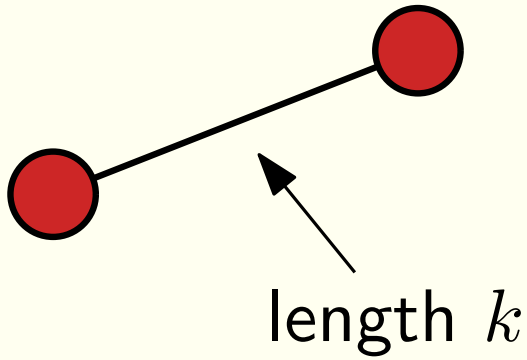

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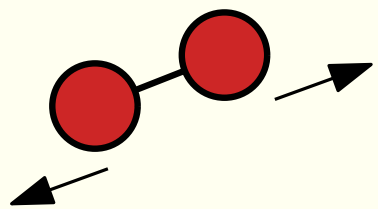

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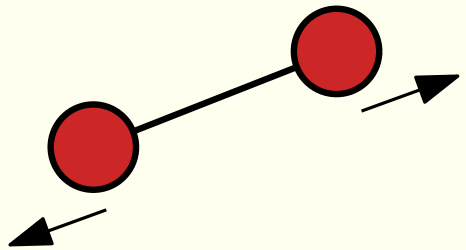

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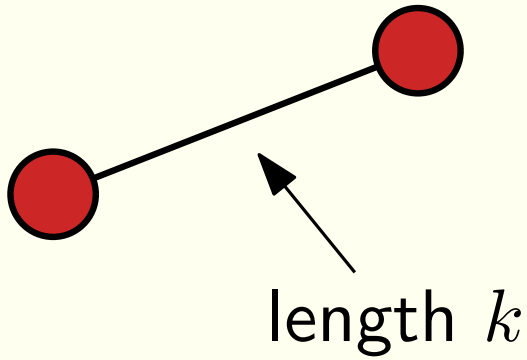

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Motivation - Graph Drawing

Many problems only have numerical algorithms

- Fruchterman-Reingold
- Kamada-Kawai
- Spectral methods
- Circle packings

Why?

Motivation - Graph Drawing

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Why?

- Galois theory!

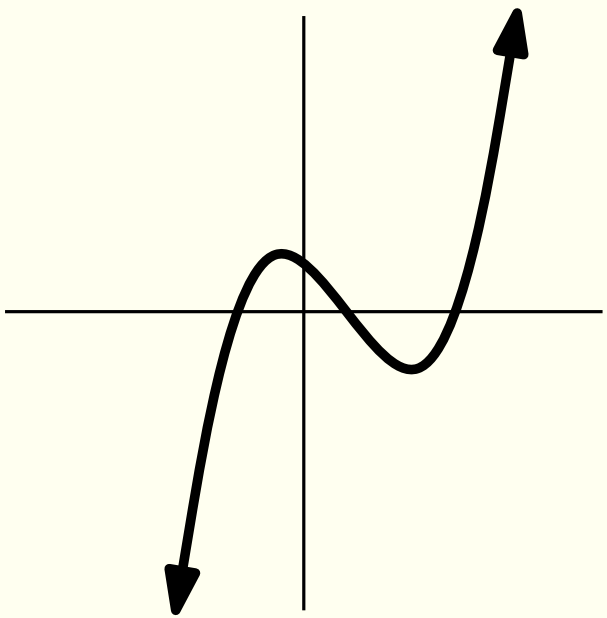
Solving polynomials

Quadratics

$$ax^2 + bx + c = 0 \quad \Rightarrow \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Cubics

$ax^3 + bx^2 + cx + d = 0 \quad \Rightarrow$ Substitute $x = t - \frac{b}{3a}$



$$t^3 + pt + q = 0$$

Substitute $t = w - \frac{p}{3w}$

$$w^6 + qw^3 - \frac{p^3}{27} = 0$$

Quadratic in w^3

Solving polynomials

Quartic

$ax^4 + bx^3 + cx^2 + dx + e = 0 \Rightarrow$ Still has a symbolic solution

Very messy

Quintic

$ax^5 + bx^4 + cx^3 + dx^2 + ex + f = 0 \Rightarrow$?

Galois - a short biography

Born in France in 1811

Mathematician

- First to use *group* as a technical term
- Worked on polynomial equations

Political activist

- Was expelled for his political opinions
- Imprisoned for threatening the King's life

Is shot and killed in the duel in 1832

Showed there is no quintic formula the night before

Galois Theory

Draws a connection between groups and roots of polynomials where the group encodes the expressibility of the roots

If the Galois group for a polynomial contains S_5 as a subgroup, then the roots cannot be written using radicals

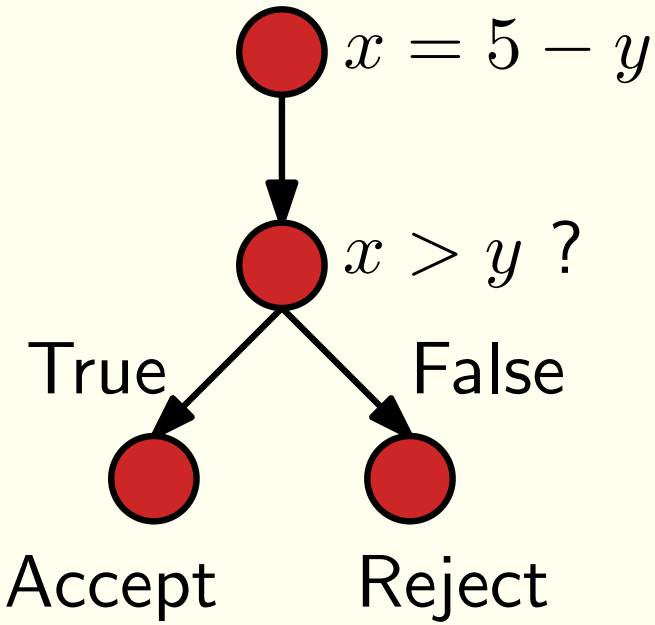
Written using radicals?

$\pi = 3.14159\ldots$ $\phi = 1.618... = \frac{1+\sqrt{5}}{2}$

Models of Computation

Algebraic computation tree

A model in which each node makes a decision or computes a value using standard arithmetic functions of previous values



Models of Computation

Quadratic computation tree

Radical computation tree

Bounded degree root computation tree

Models of Computation

Quadratic computation tree

An algebraic computation tree with square roots and complex conjugation

Radical computation tree

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Radical computation tree

An algebraic computation tree with k^{th} roots and complex conjugation for any integer k

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Bounded degree root computation tree

An algebraic computation tree with taking roots of bounded degree polynomials and complex conjugation

Models of Computation

Quadratic computation tree

An algebraic computation tree with square roots and complex conjugation

Compass and straightedge model

Radical computation tree

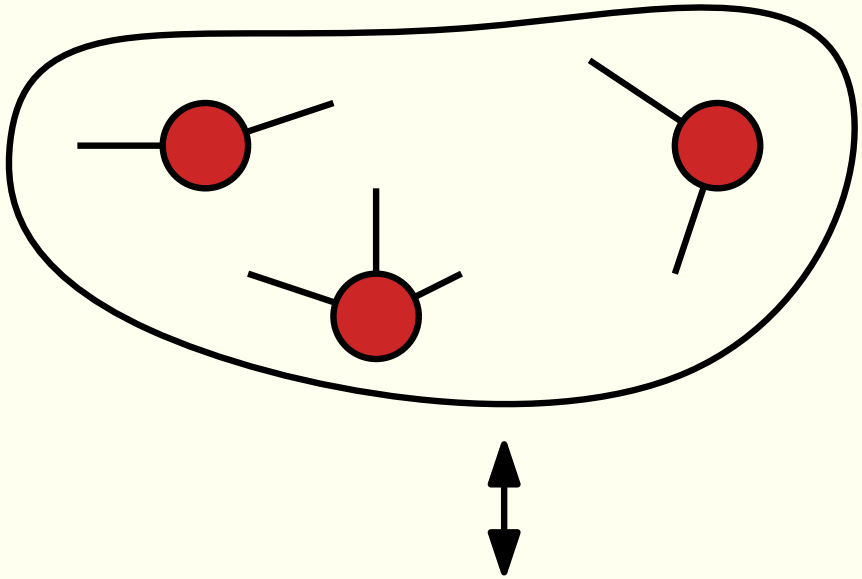
An algebraic computation tree with k^{th} roots and complex conjugation for any integer k

Bounded degree root computation tree

An algebraic computation tree with taking roots of bounded degree polynomials and complex conjugation

Approach

F & R Layout



Graph Drawing Layout

Polynomial

Galois Group

Expressibility in a Symbolic Model

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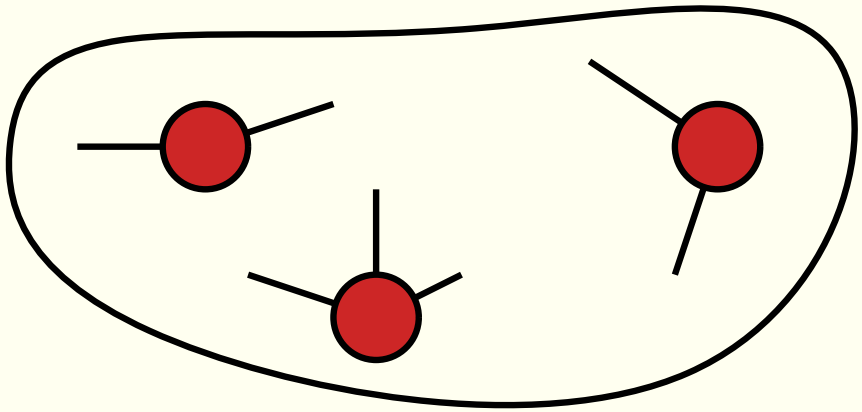
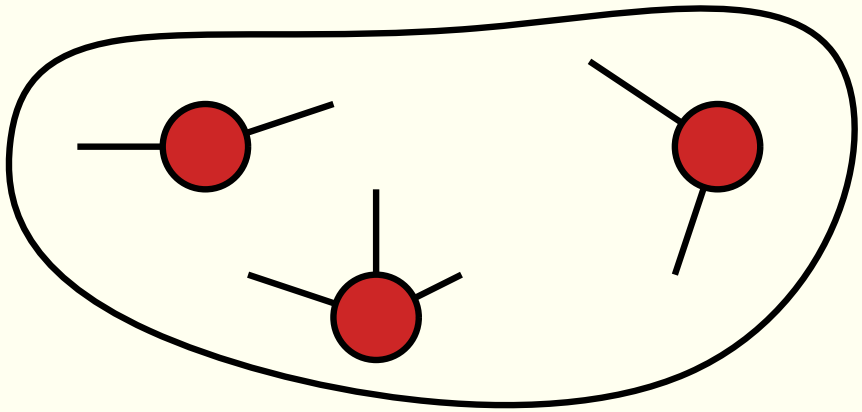


$$p(x) = x^n + \ldots$$

Approach

Graph Drawing Layout



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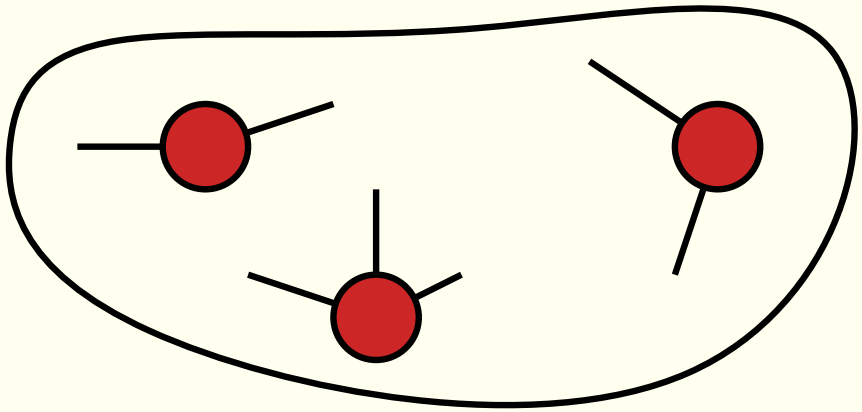

F & R Layout

Polynomial

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S_5

Expressibility in a Symbolic Model

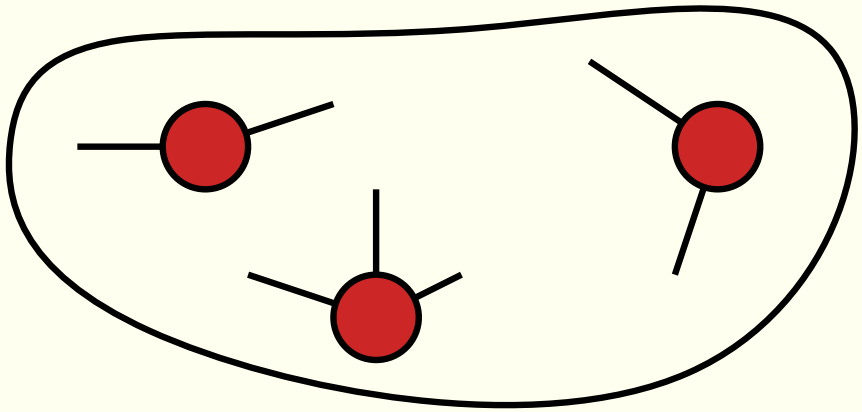
Cannot draw in a Radical Computation Tree

Approach

F & R Layout

Lots of variables

System of polynomials



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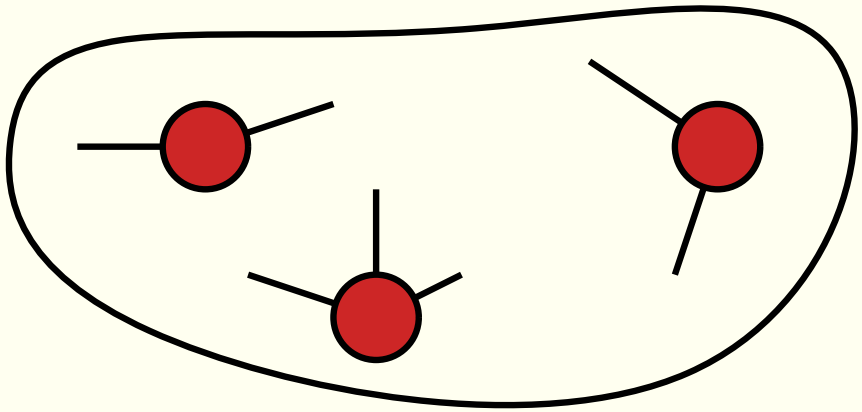

Lots of variables

System of polynomials

$\Rightarrow p(x)$ may have high degree

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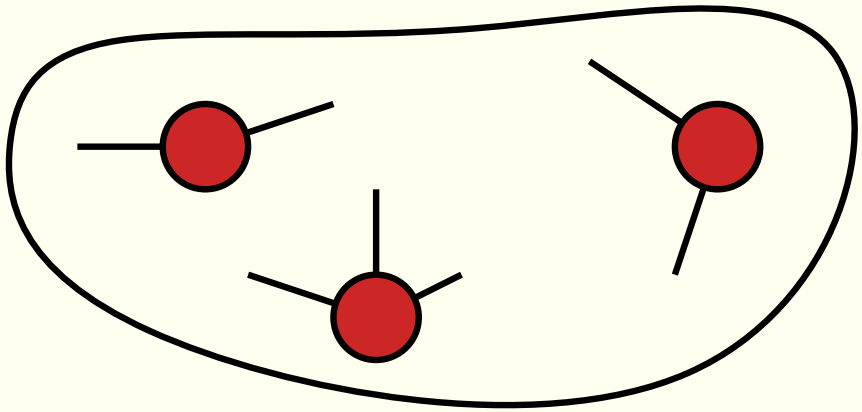

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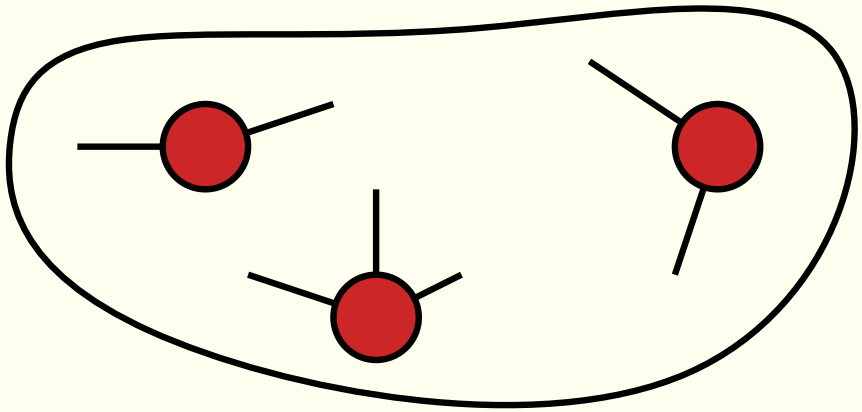


Lots of variables

System of polynomials

$\Rightarrow p(x)$ may have high degree

Exploit symmetry to reduce degree

$$p(x) = x^n + \ldots$$

$$S_5$$

Cannot draw in a Radical Computation Tree

Undrawable Graphs

Bounded degree root computation trees

Fruchterman-Reingold

Kamada-Kawai

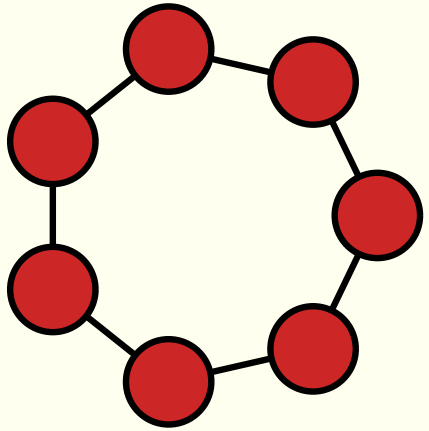
Spectral graph drawings

Circle packings

Undrawable Graphs

Bounded degree root computation trees

p-cycles



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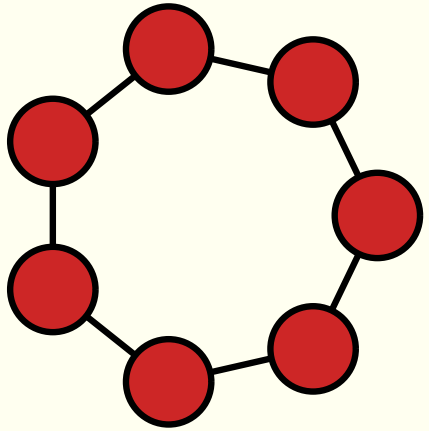
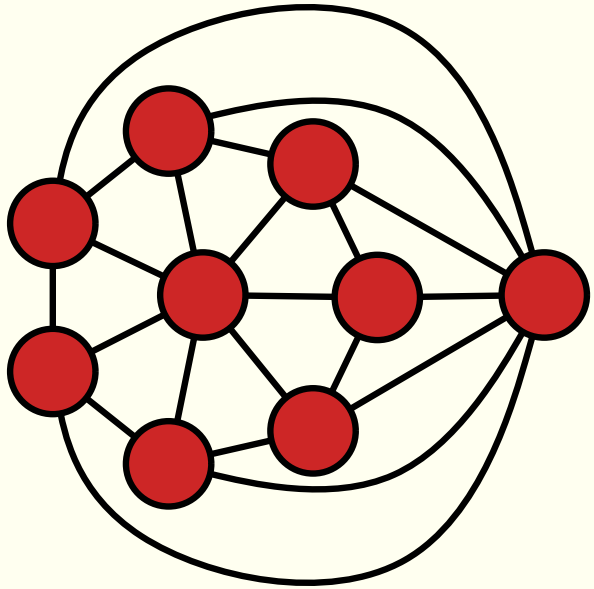


p-bipyramid



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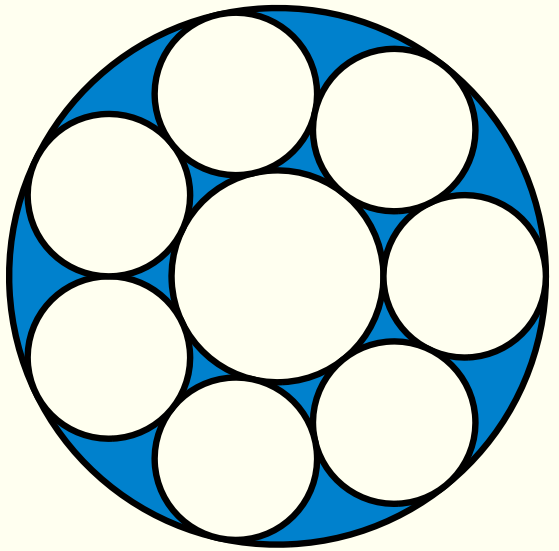

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Multidimensional scaling

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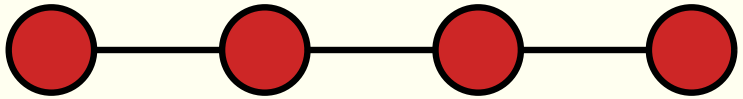

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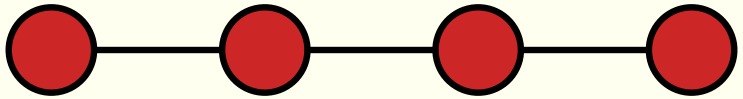

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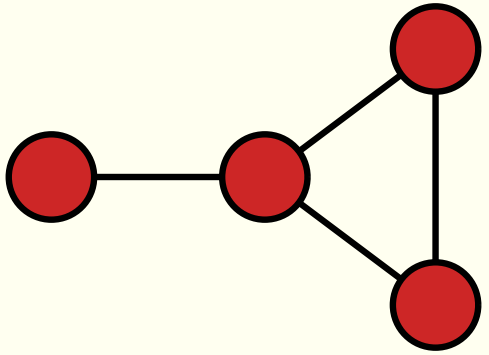

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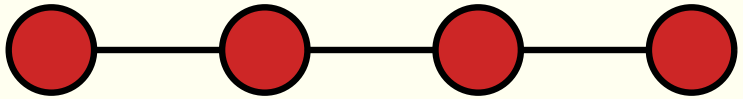

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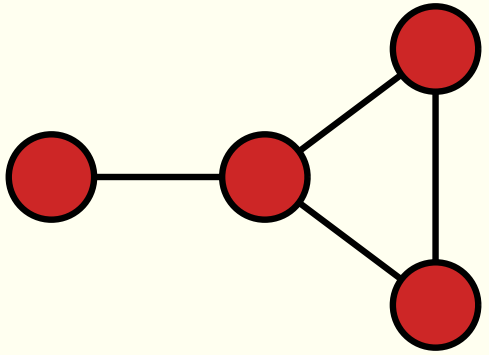

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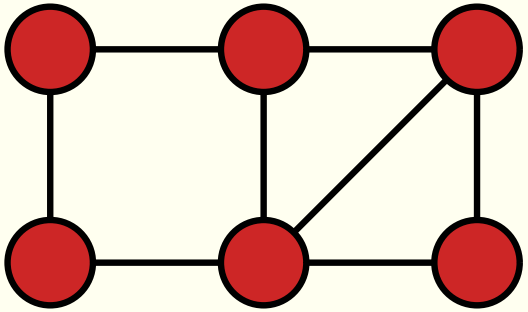

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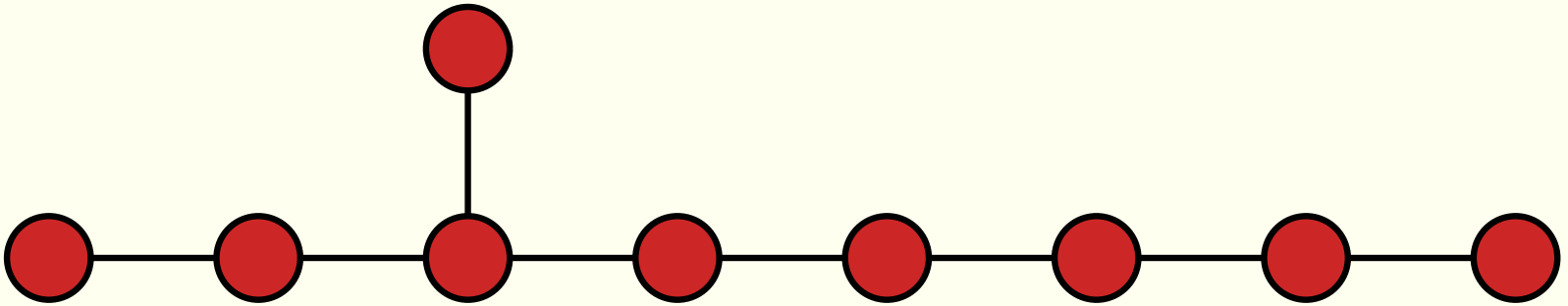
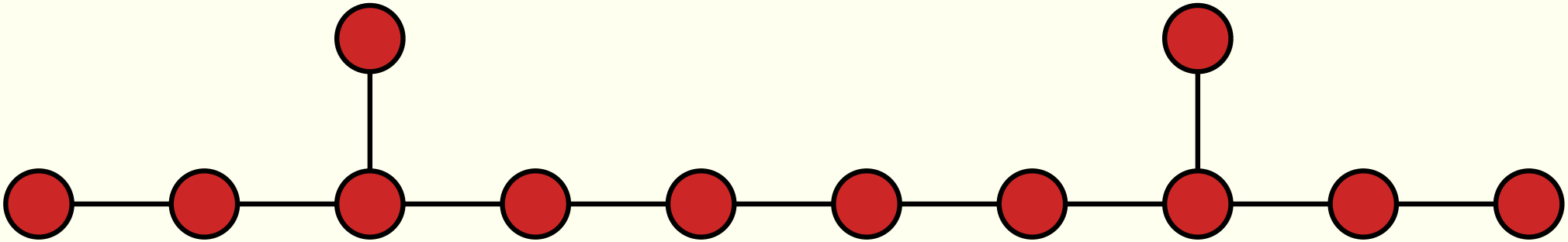


(Laplacian matrix)



(Adjacency/Transition matrix)

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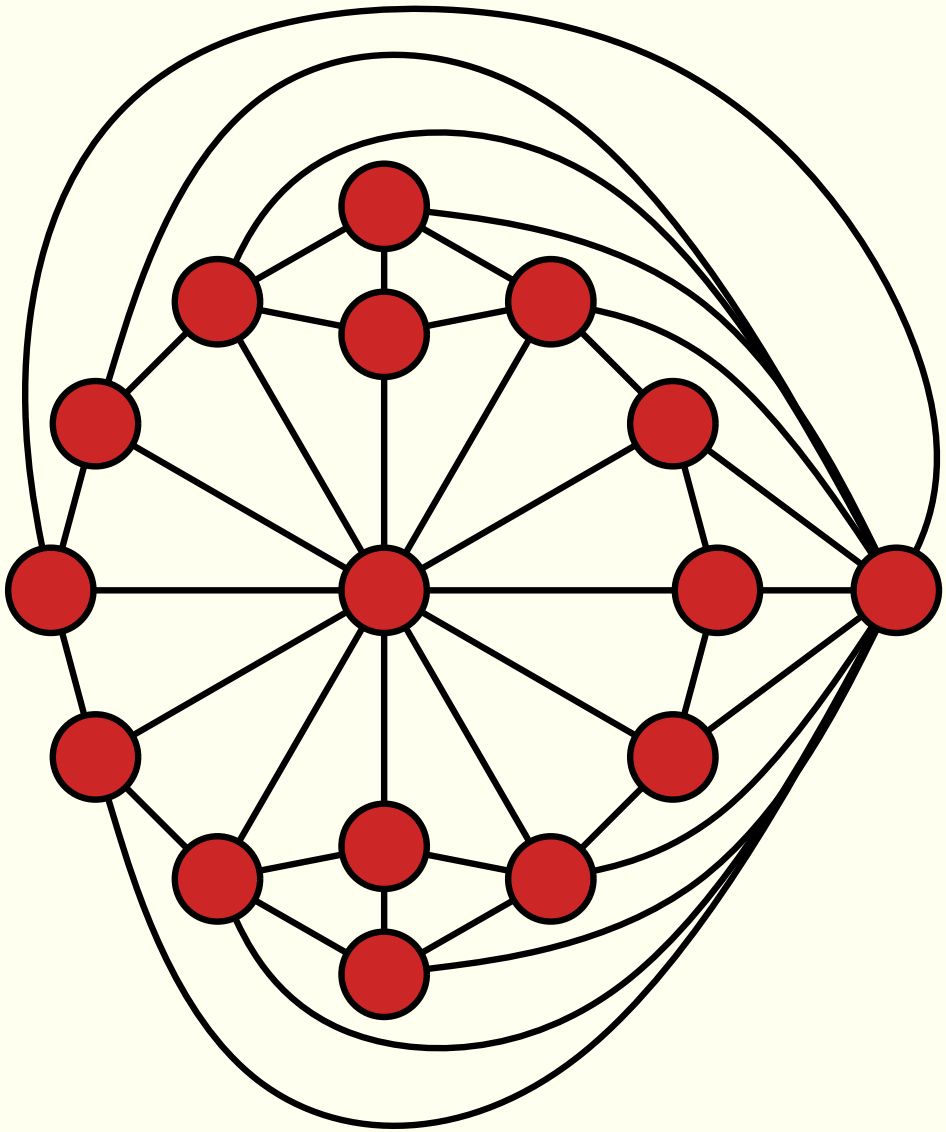

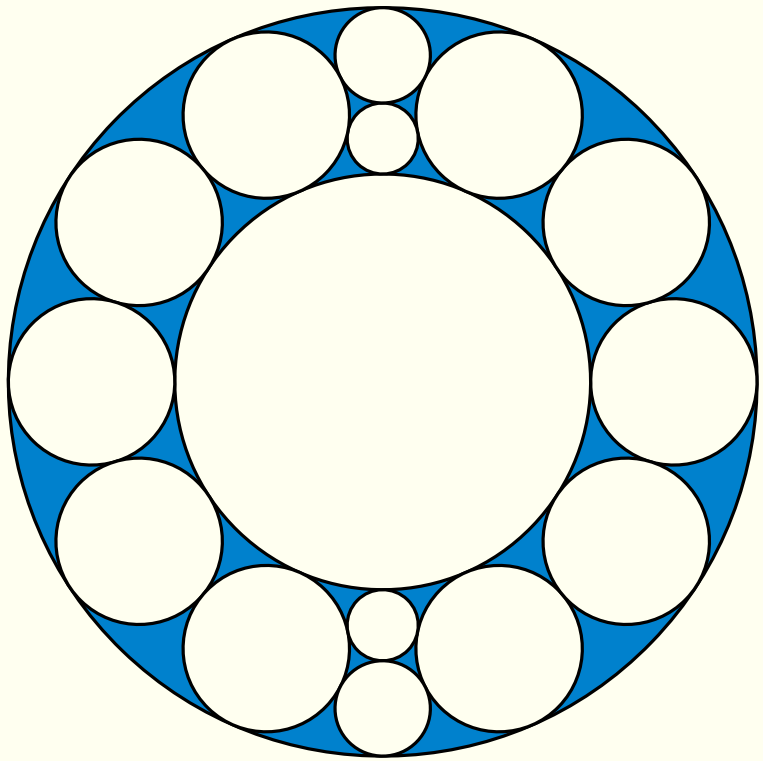

Summary

Lots of graph drawing uses numerical algorithms

Why no symbolic algorithms? Galois theory!

Graph drawing coordinates cannot be computed using radicals

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Why no symbolic algorithms? Galois theory!

Graph drawing coordinates cannot be computed using radicals

Open Questions

Other graph drawing problems with no symbolic algorithms?

Problems with arbitrarily high S_n?

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Thank you