Crossing Minimization for 1-page and 2-page Drawings of Graphs with Bounded Treewidth

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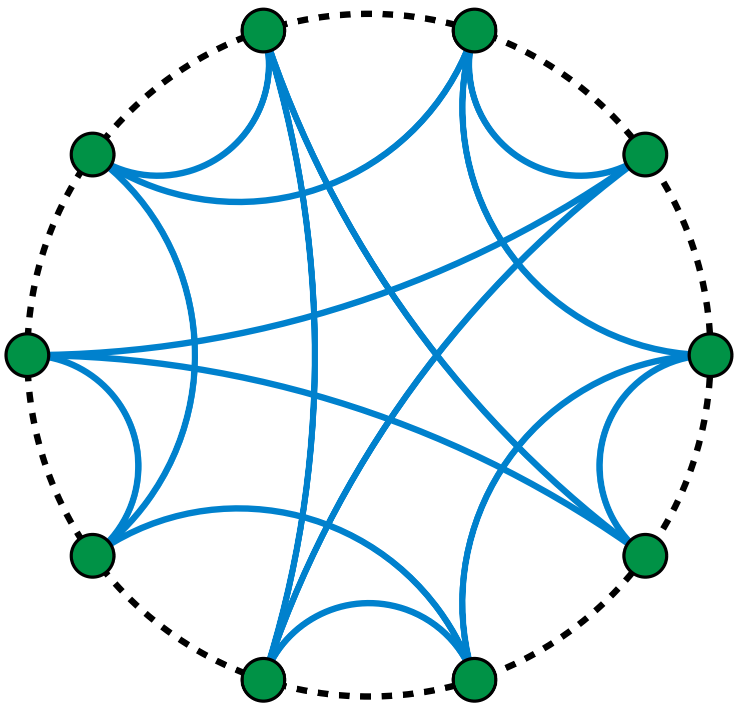
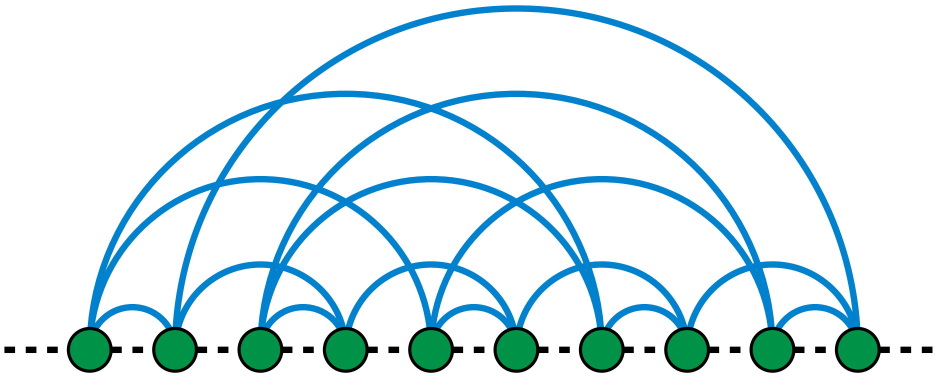
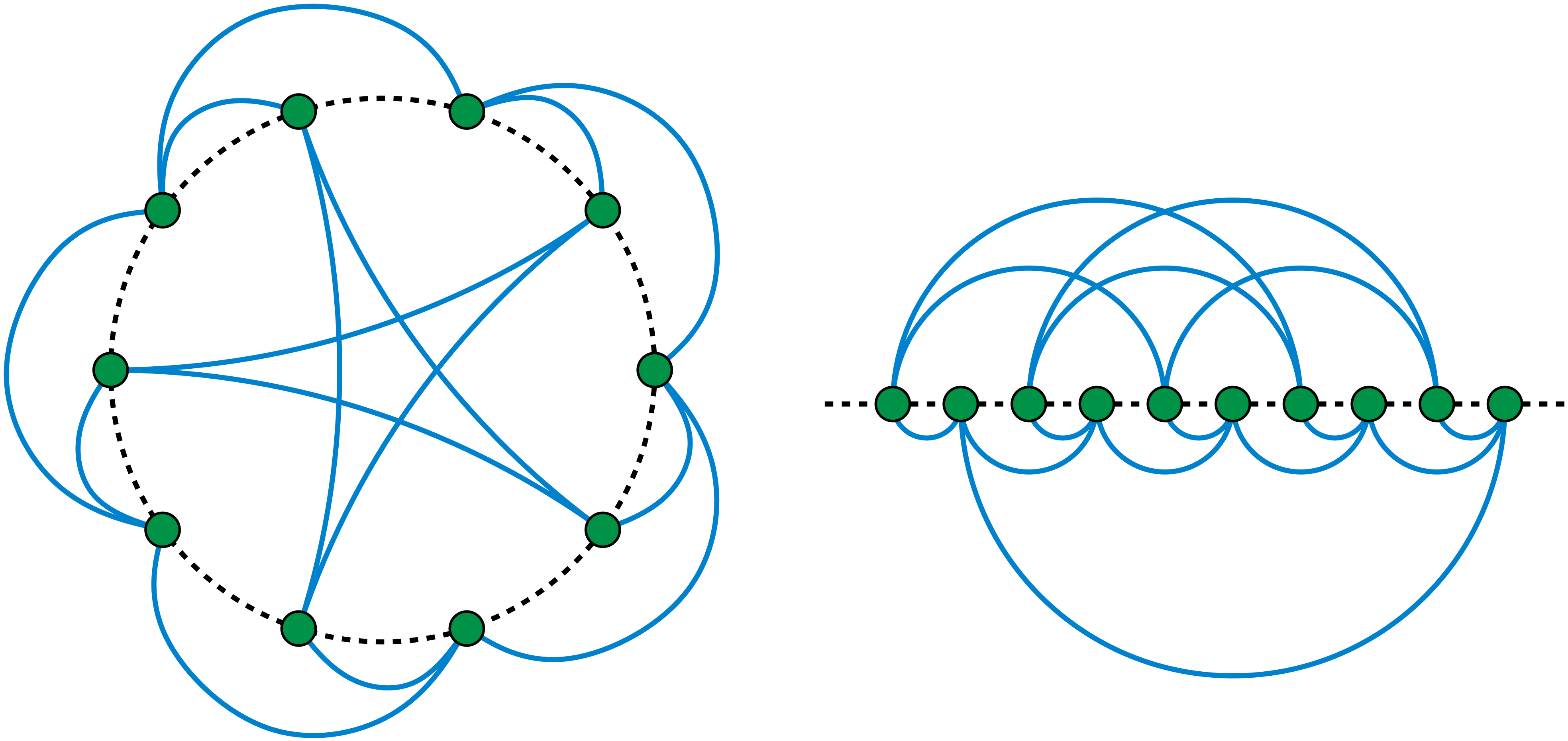
Book drawing: 1-Page

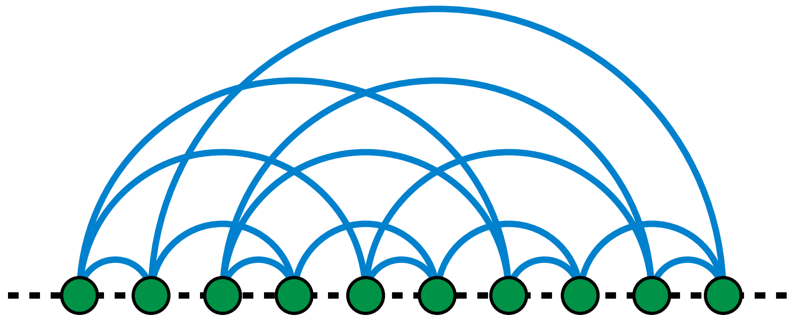


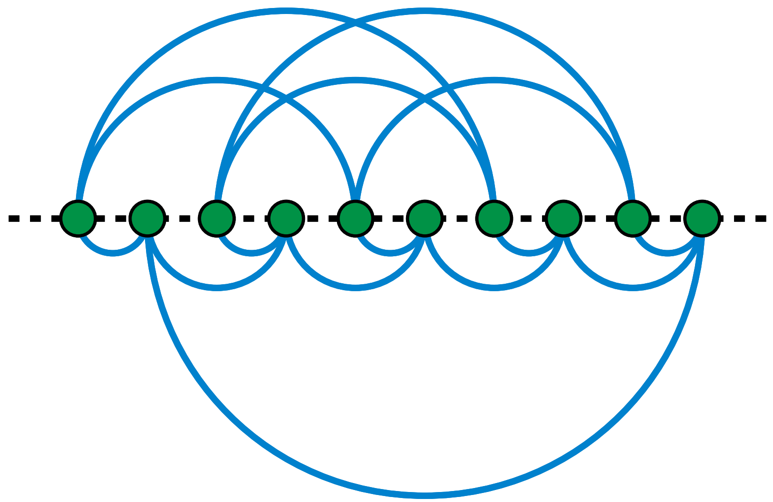


Book drawing: 2-Page



Crossing minimization





1-Page

- Min is NP-Hard
- Planar in P

2-Page

- Min is NP-Hard
- Planar is NP-Hard
- Fixed vertex NP-Hard

Fixed-paramter tractability

NP-hard implies runtime is likely (at least) exponential

Exponential in what?

Maybe some parameter less than input

Goals:

- Find a small parameter p of the inputs
- Find an algorithm running in $O(f(p)n^c)$
- f must be computable and $c = O(1)$

If achieved, then the problem is fixed-parameter tractable

Previous results

GD 2013 results (B, Eppstein, Simons):

Crossing

1-Page $O((5k)!^{\omega/3}n)$

2-Page $O(2^{6k^3}(6k^3)!^{\omega/3}n)$

Crossed edges

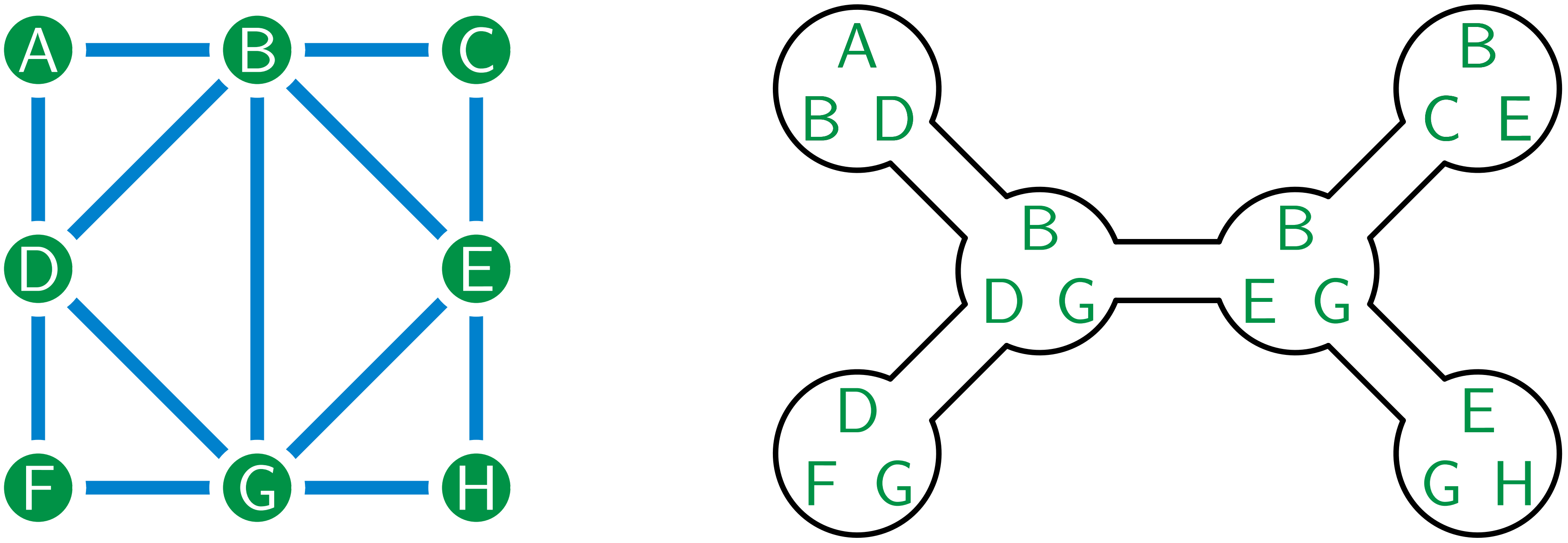
1-Page $O((5k)!^{\omega/3}n)$

2-Page $O(2^{6k^2}(6k^2)!^{\omega/3}n)$

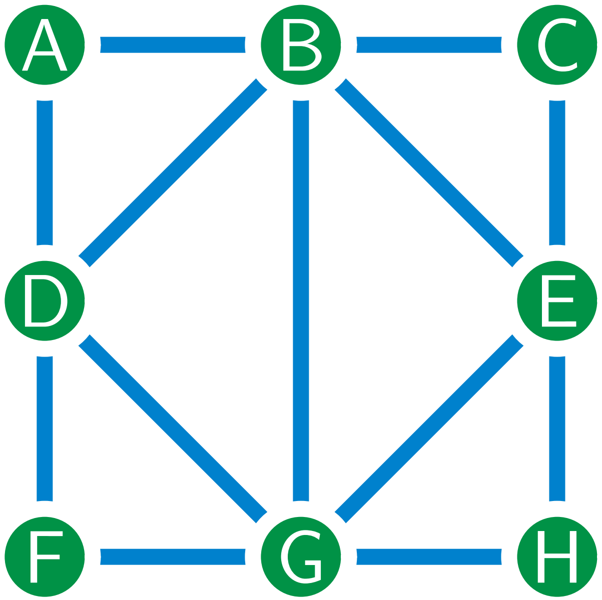
$k =$ cyclomatic number or almost-tree parameter

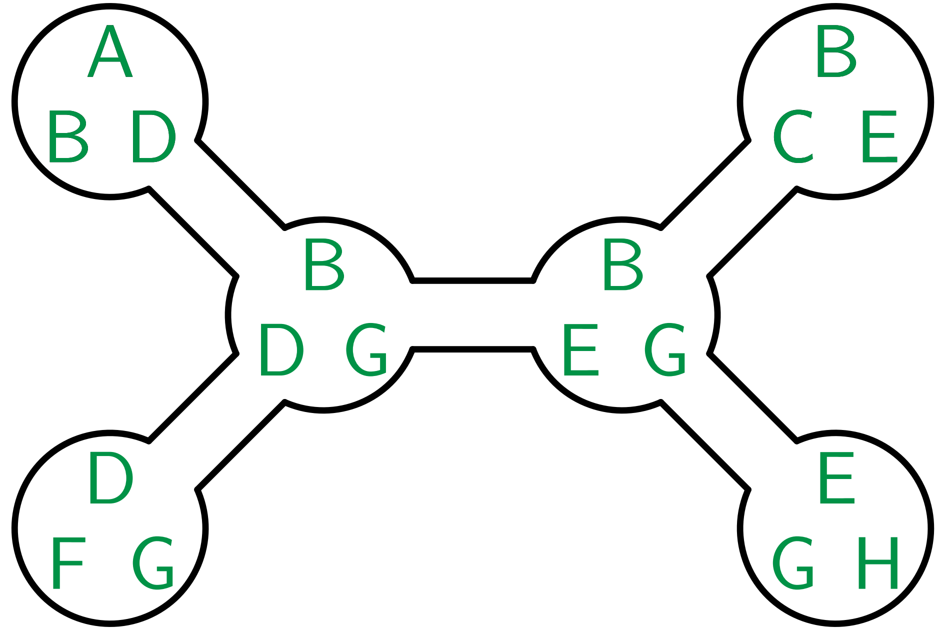
$\omega =$ exponent of matrix multiplication

Treewidth



Treewidth

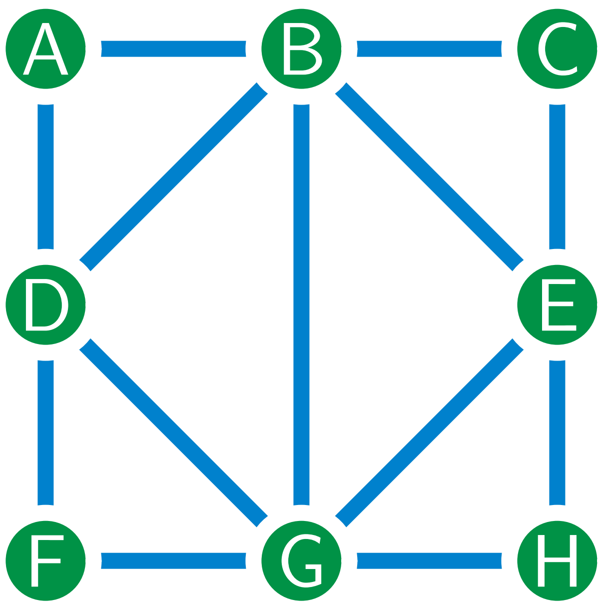


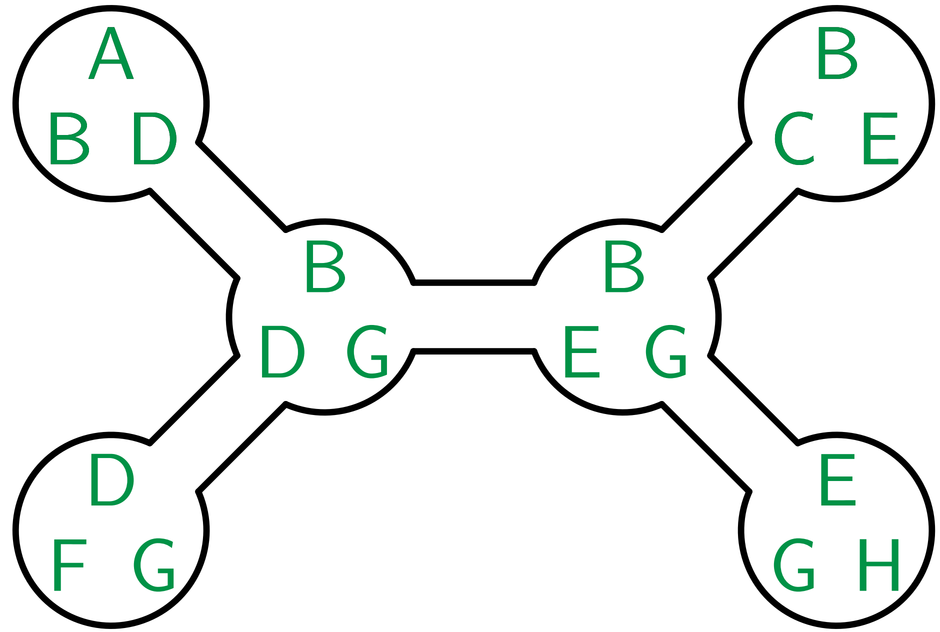


Tree decomposition

- Tree with nodes subsets of V called bags
- Every $v \in V$ is in some bag
- $v \in B_1 \cap B_2 \Rightarrow v \in B \ \forall B$ on the path from B_1 to B_2.
- $uv \in E \Rightarrow u, v \in B$ for some bag B

Treewidth





Width of the decomposition

- One less than the size of largest bag

Treewidth

- Width of the smallest decomposition

Computability

- NP-Complete, but FPT $O\big(f(k)n\big)$

Monadic second order logic (MSO_2)

Vertex and edge variables: $v_0, v_1, \ldots$, $e_0, e_1, \ldots$

Vertex and edge set variables: $V_0, V_1, \ldots, E_0, E_1, \ldots$

Binary relations: $=$, $\in$, I

Propositional logic operations: $\neg, \wedge, \vee, \rightarrow$

Quantifiers: $\forall$, $\exists$

Examples of properties expressible in MSO_2

k-coloring	minor containment
connectedness	planarity
hamiltonicity	outerplanarity

Courcelle's theorem

Input: A graph G and an MSO_2-formula ϕ

Parameter: treewidth(G) + length(ϕ)

Output: Does G satisfy ϕ

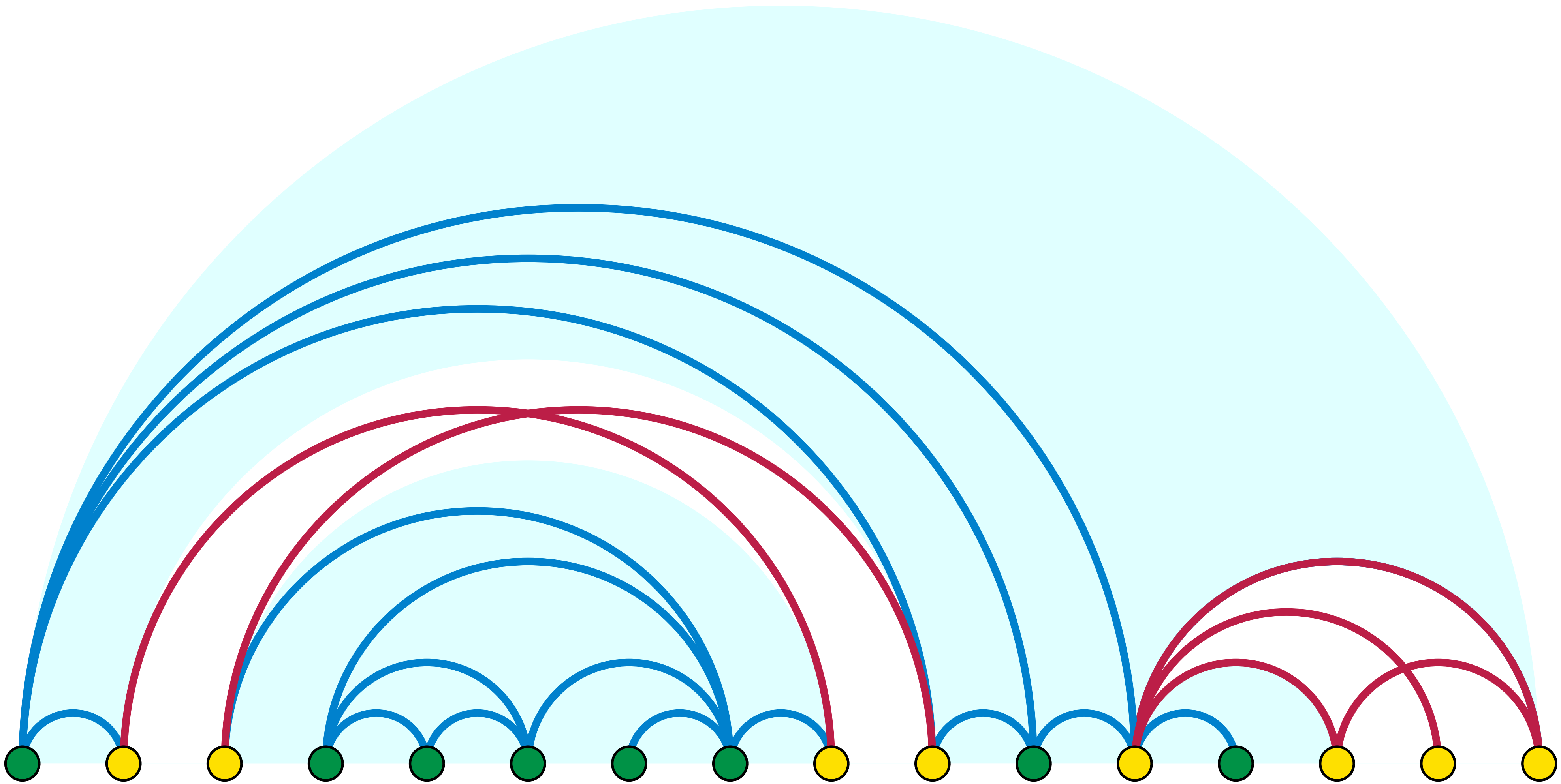
Runtime: $f(k, \ell)n$

Application:

Crossing minimization is FPT in the $\#$ crossings

Grohe (2001), Kawarabayashi & Reed (2007)

1-Page drawing decomposition



1-Page formula construction

For each crossing configuration D with k crossings:

$\alpha(v_0, \ldots, e_0, \ldots) =$ fixes crossing edges/vertices

$\beta(U_0, U_1, \ldots) =$ no edges across regions

$\gamma(U_0, U_1, \ldots) =$ each region is "outerplanar"

$\delta_D = (\exists v_0, \ldots)(\exists e_0, \ldots)(\exists U_0, \ldots)[\alpha \wedge \beta \wedge \gamma]$

$\mathrm{ONEPAGE}_k = \bigvee_D \delta_D$

length$(\mathrm{ONEPAGE}_k) = 2^{O(k^2)}$

1-Page crossing min is FPT !

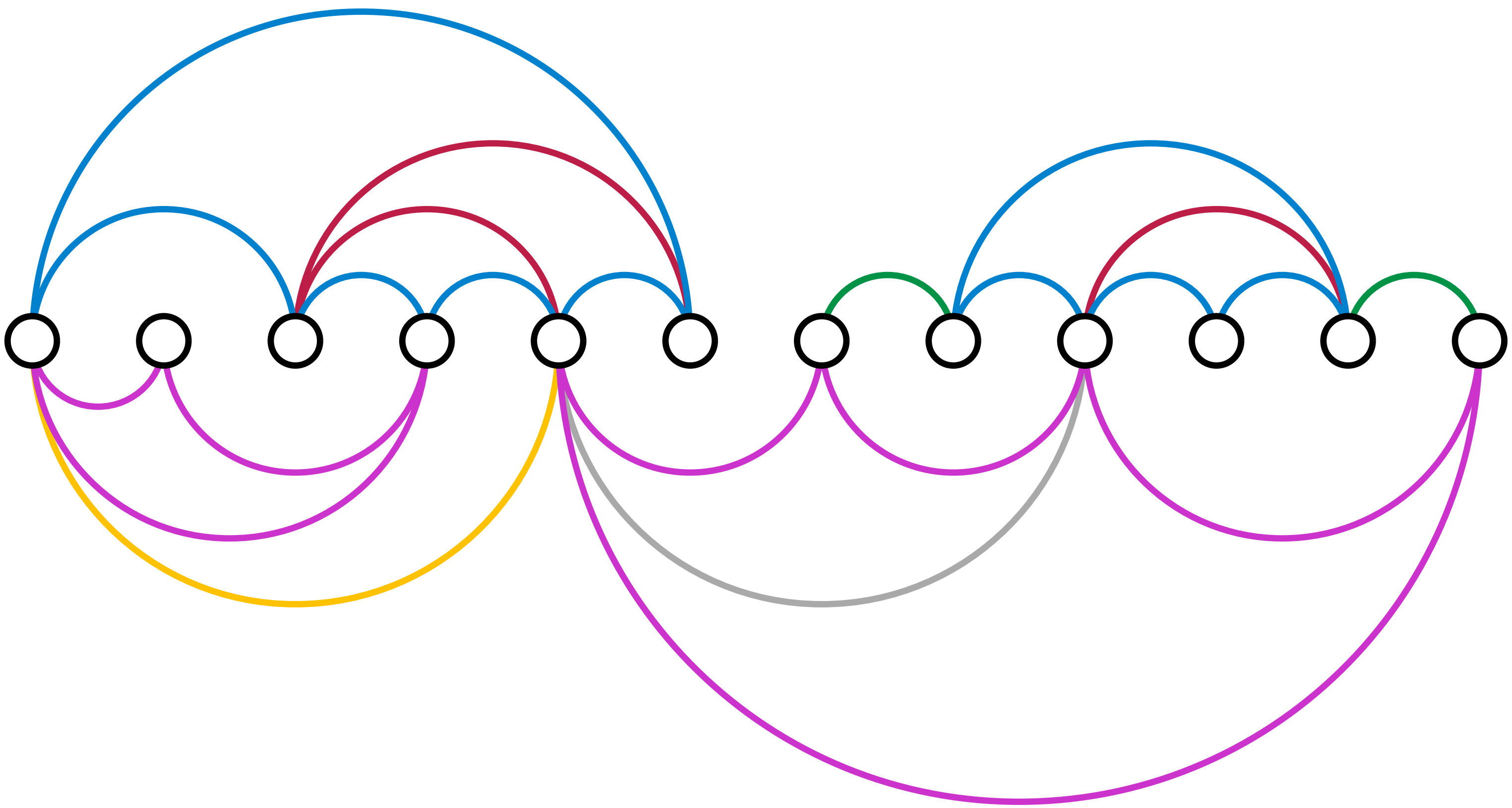
2-Page planarity idea

Subgraph of planar + Hamiltonian?

No subgraph in MSO_2.

$\exists$ a part of the edges into two outerplanar graphs?

Different vertex orders.



Our results

- 1-Page crossing minimization is FPT
  parameter: crossing number
- 2-Page planarity is FPT
  parameter: treewidth
- 2-Page crossing minimization is FPT
  parameter: treewidth + crossing number

Open problems

- Can the dependency on the parameters be reduced?
- Is 2-page crossing NP-hard for fixed treewidth?
- Can book thickness k be expressed in MSO_2?

Thank You!