The Importance of Being Proper

(In Clustered-Level Planarity and T-Level Planarity)

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joint work with

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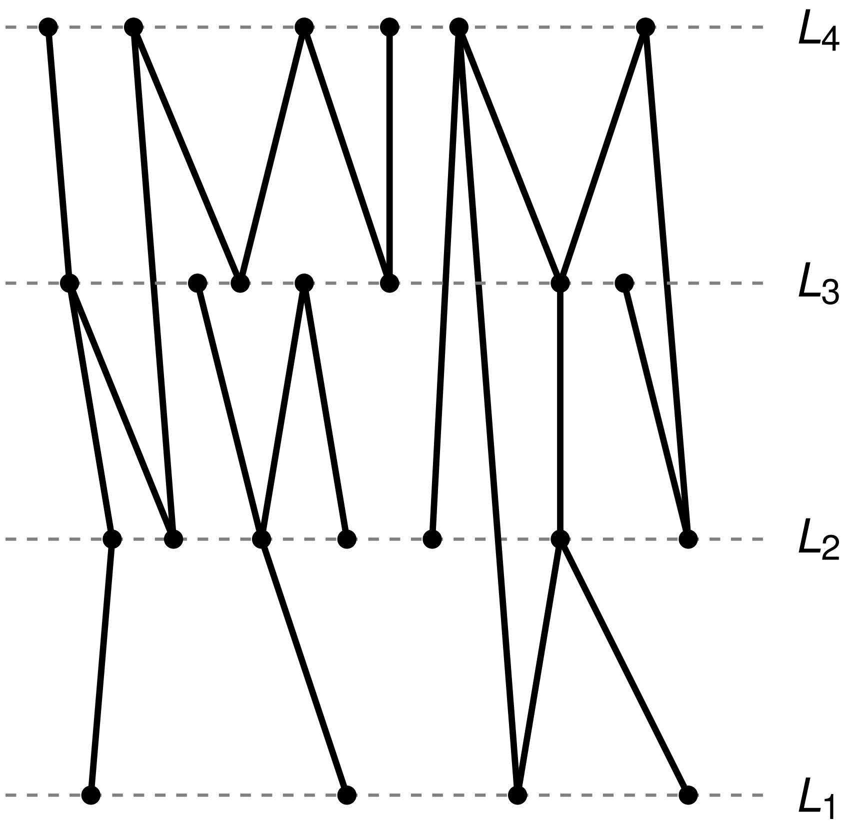
Roman Schneidmüller (photo)

Level Planarity

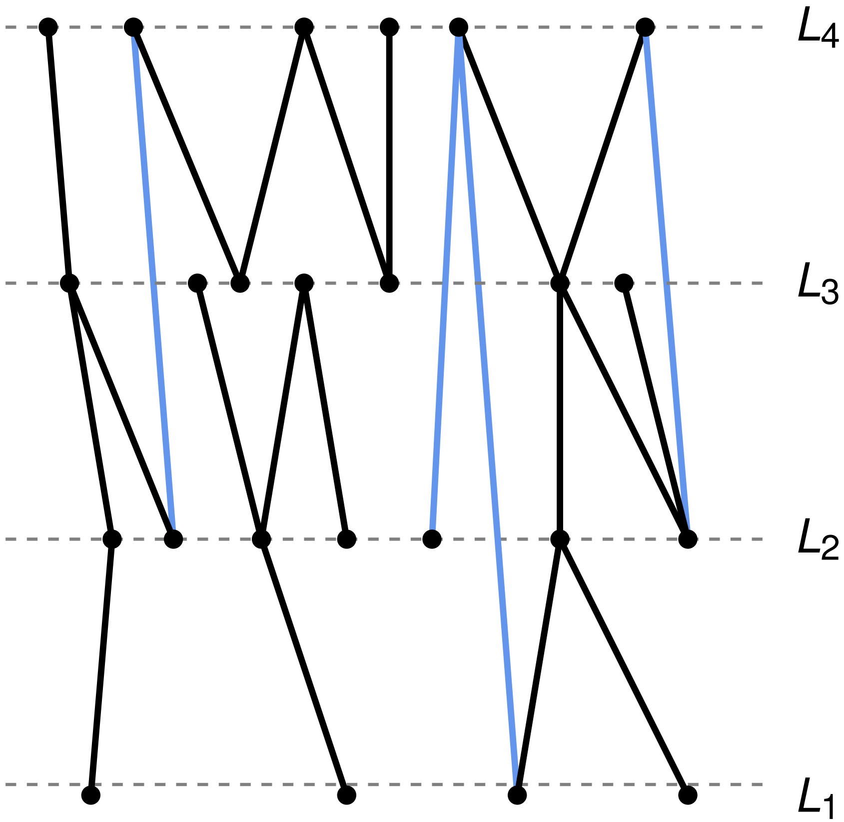


$$(V, E, \gamma), \gamma : V \to \{1, 2, ..., k\}$$

Theorem [Jünger, Leipert, and Mutzel - GD'98]

$O(|V|)$-time testing algorithm

Proper Level Graphs



$$\forall(u, v) \in E : \gamma(u) = \gamma(v) \pm 1$$

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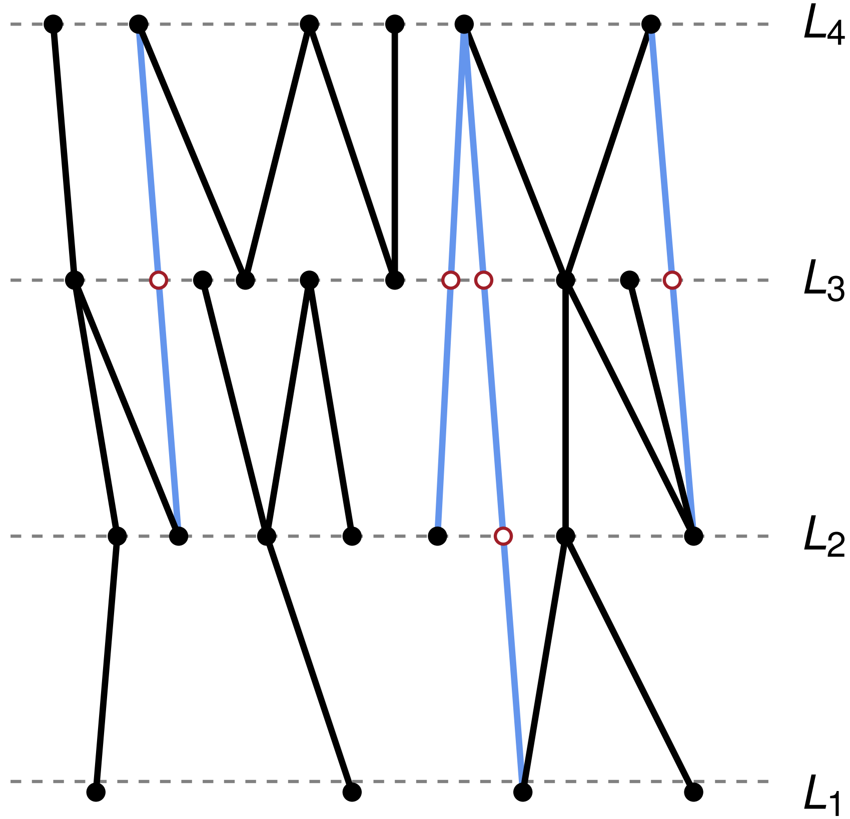
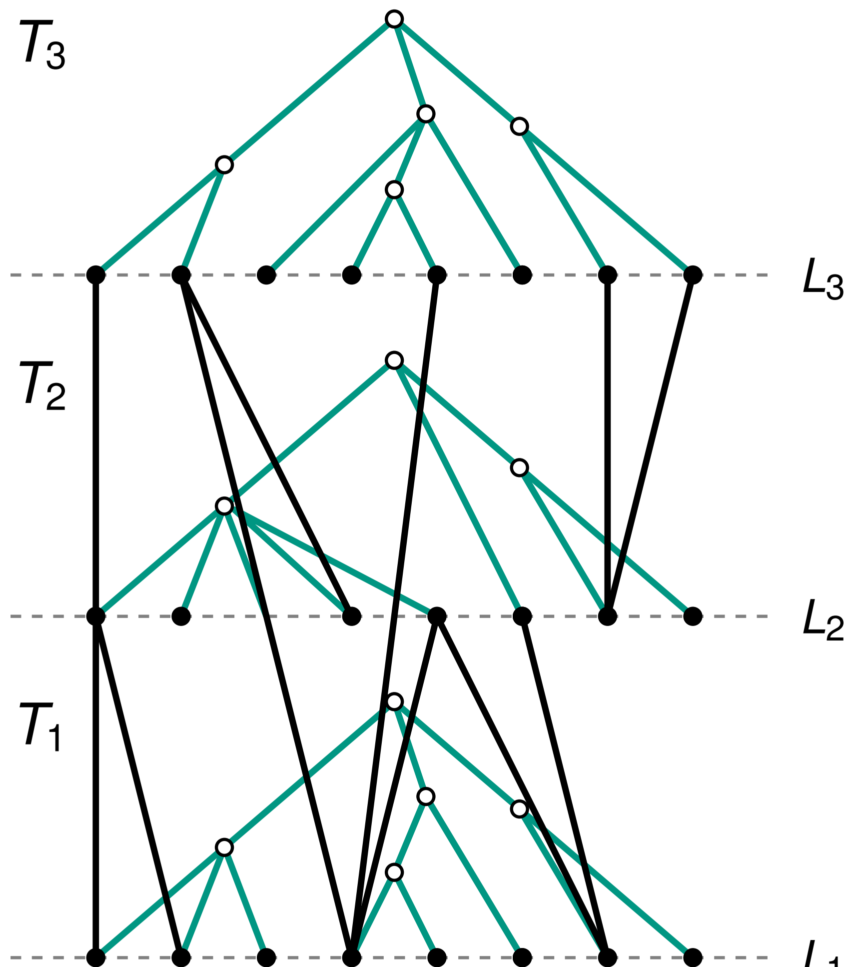


$$\forall (u, v) \in E : \gamma(u) = \gamma(v) \pm 1$$

Common assumption:

if the input graph is not proper, then we can make it proper by "simply adding dummy vertices"

Variants of L-Planarity: T-LEVEL PLANARITY



$(V, E, \gamma, \mathcal{T}), \mathcal{T} = \{T_1, \ldots, T_k\}$

Theorem [Wotzlaw, Speckenmeyer, and Porschen - DAM'12]

$O(|V|^2)$-time algorithm if (V, E, γ) is **proper** and $max_i(|V_i|)$ is **bounded** by a constant

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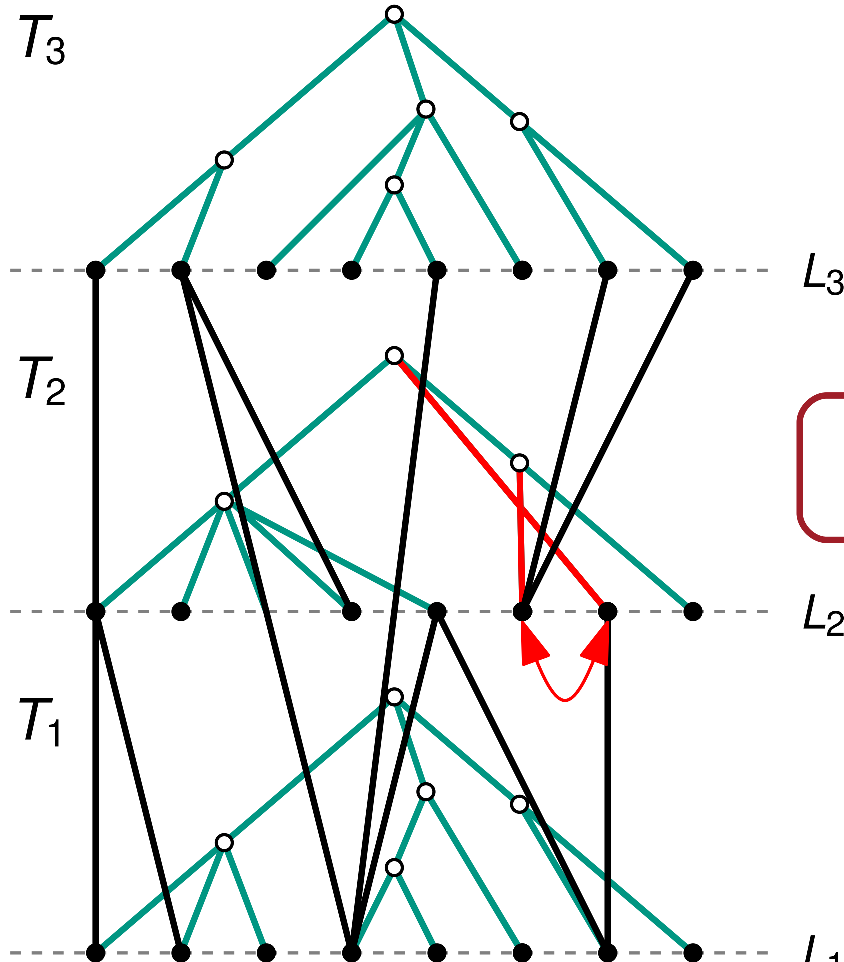

The ordering along L_2 is **not compatible with** T_2

$(V, E, \gamma, \mathcal{T}), \mathcal{T} = \{T_1, \dots, T_k\}$

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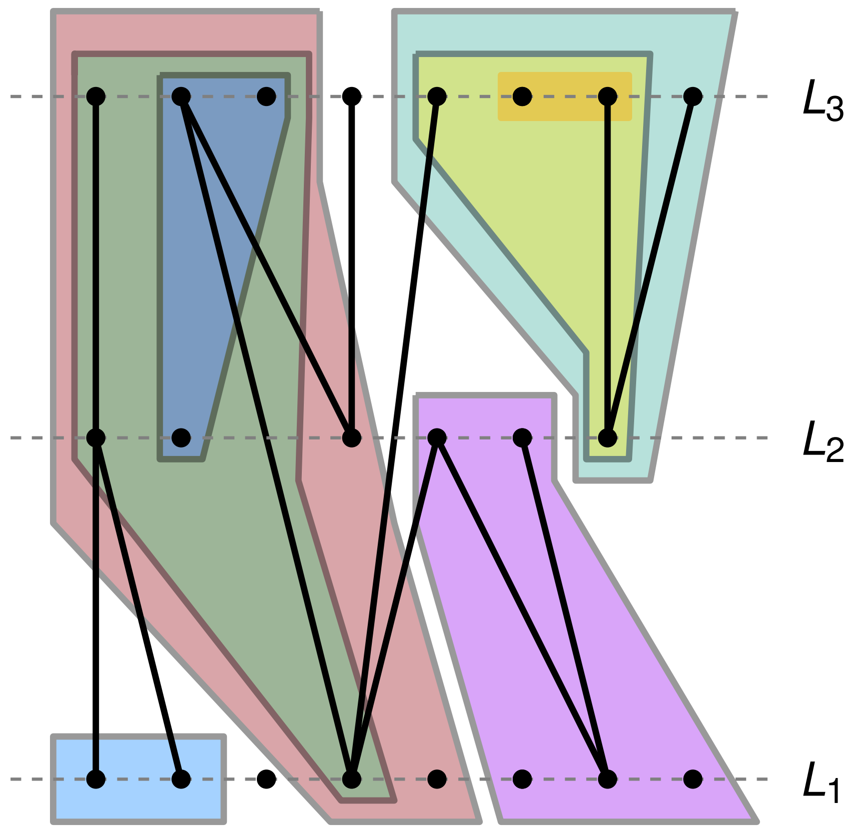
Variants of L-Planarity: CL-PLANARITY

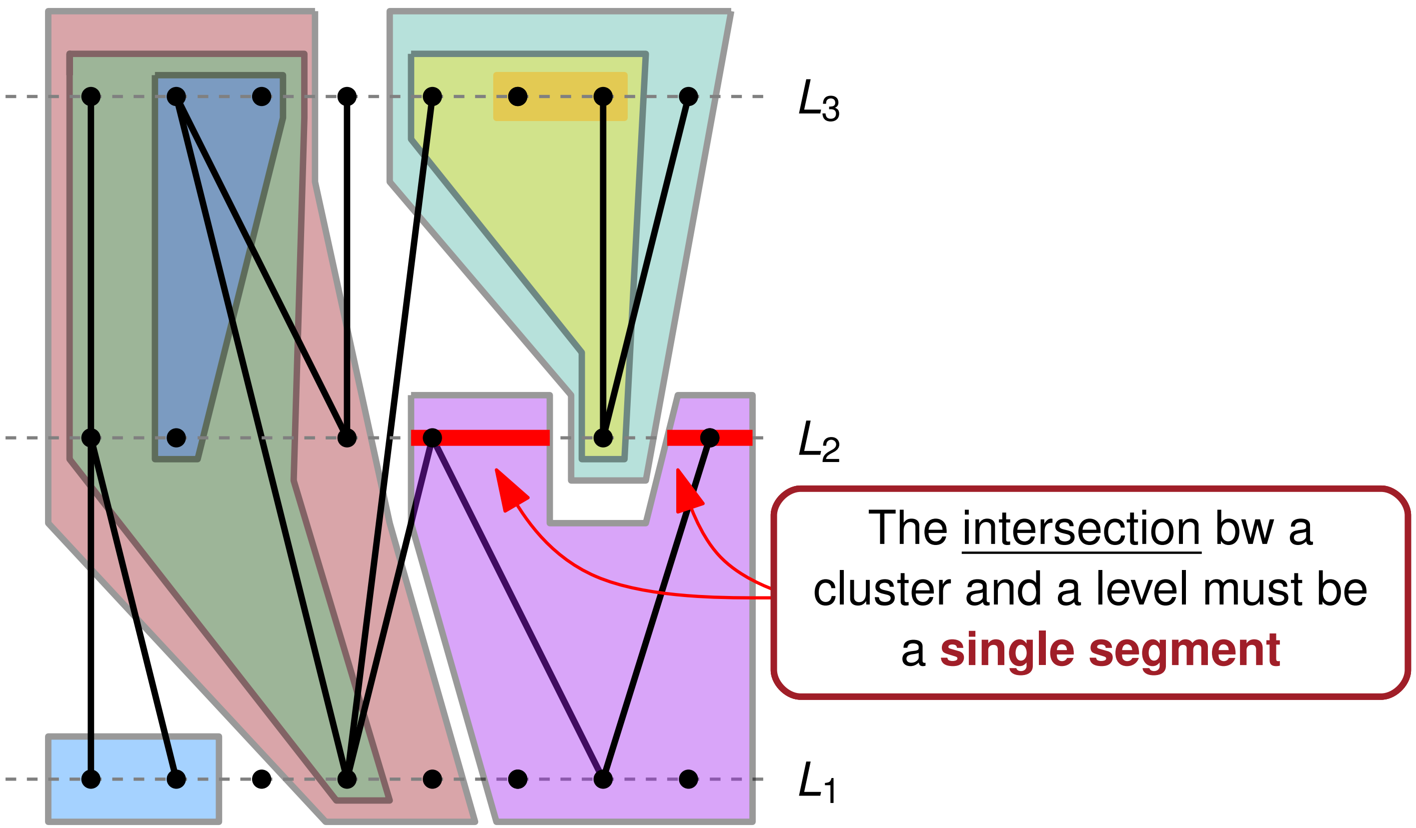


(V, E, γ, T), Inclusion Tree T

Theorem [Forster and Bachmaier - SOFSEM'04]

$O(k|V|)$-time algorithm if (V, E, γ) is a **proper hierarchy** and clusters are **level-connected**

Variants of L-Planarity: CL-PLANARITY



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Complexity results

	L-Planarity	T-Level Planarity	CL-Planarity
NON-PROPER	$O(n)$	?	?
PROPER	$O(n)$	?	?

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	L-Planarity	T-Level Planarity	CL-Planarity
NON-PROPER	$O(n)$	$\mathcal{NP}$-complete	$\mathcal{NP}$-complete
PROPER	$O(n)$	$O(n^2)$	$O(n^4)$

Complexity results: non-proper instances

	L-Planarity	T-Level Planarity	CL-Planarity
NON-PROPER	$O(n)$	$\mathcal{NP}$-complete	$\mathcal{NP}$-complete
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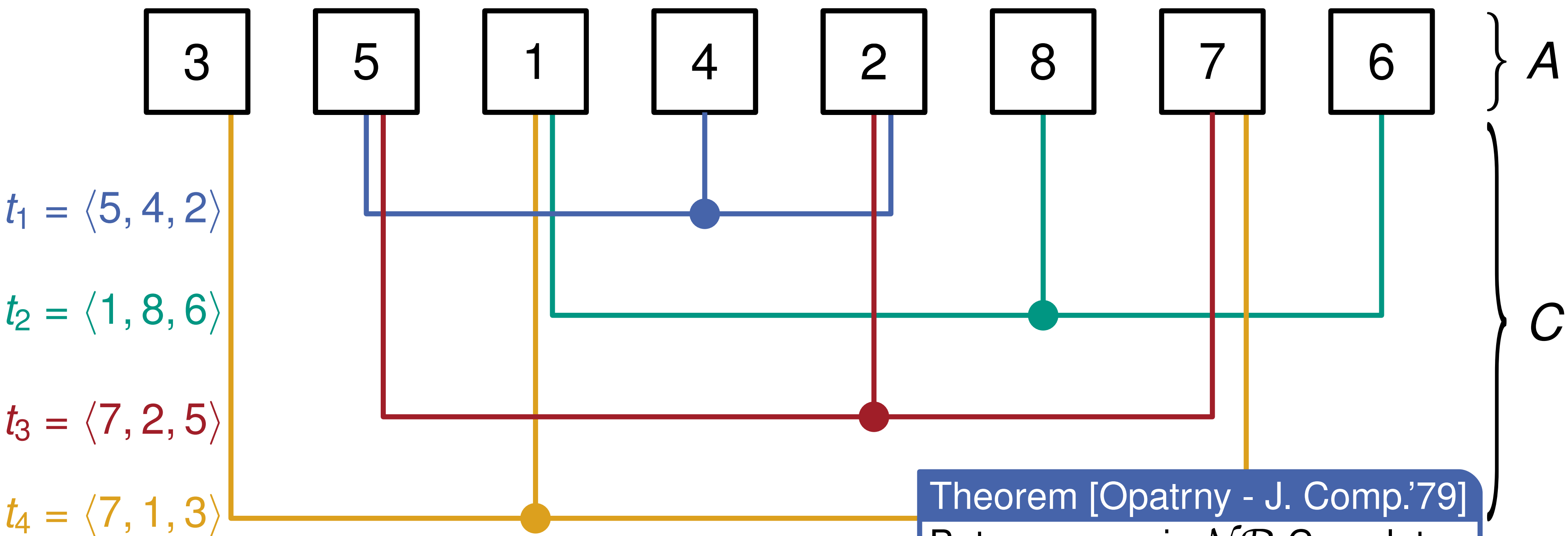
The Betweenness Problem

- **input**: pair $\langle A, \mathcal{C} \rangle$
  - a finite set A of n objects
  - a set $\mathcal{C}$ of m ordered triples $t_i = \langle \alpha_i, \boldsymbol{\beta}_i, \delta_i \rangle$ of distinct elements of A
- **question**: is there a **linear ordering** $\mathcal{O}$ of A such that, for each triple $t_i \in \mathcal{C}$, either $\mathcal{O} = \langle \ldots, \alpha_i, \ldots, \boldsymbol{\beta}_i, \ldots, \delta_i, \ldots \rangle$ or $\mathcal{O} = \langle \ldots, \delta_i, \ldots, \boldsymbol{\beta}_i, \ldots, \alpha_i, \ldots \rangle$?

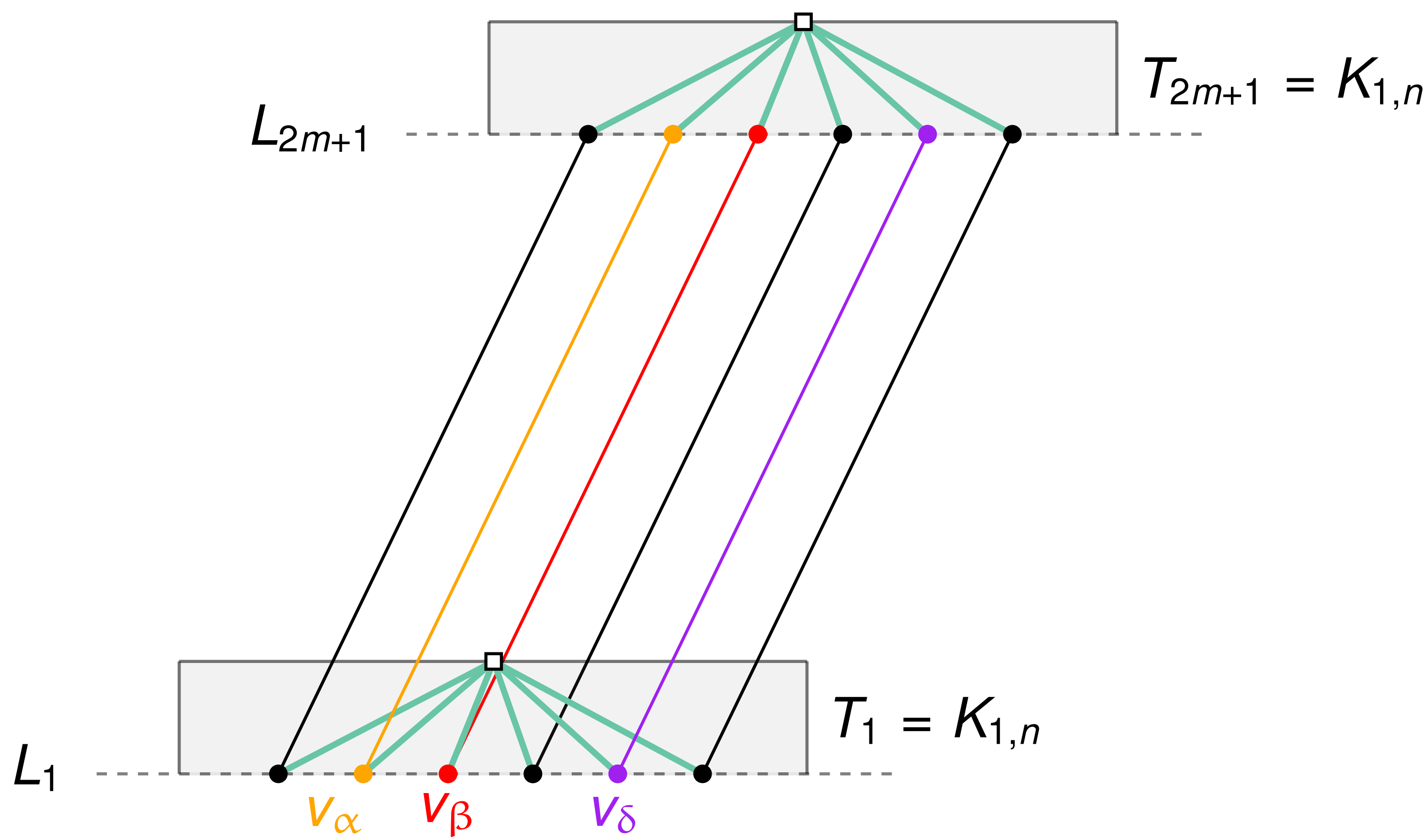


Theorem [Opatrny - J. Comp.'79]
Betweenness is $\mathcal{NP}$-Complete

T-LEVEL PLANARITY is $\mathcal{NP}$-hard

BETWEENNESS → $(V, E, \gamma, \mathcal{T})$ of T-LEVEL PLANARITY

- Graph (V, E) is a set of paths
- $\mathcal{T}$ contains $2m$ binary trees



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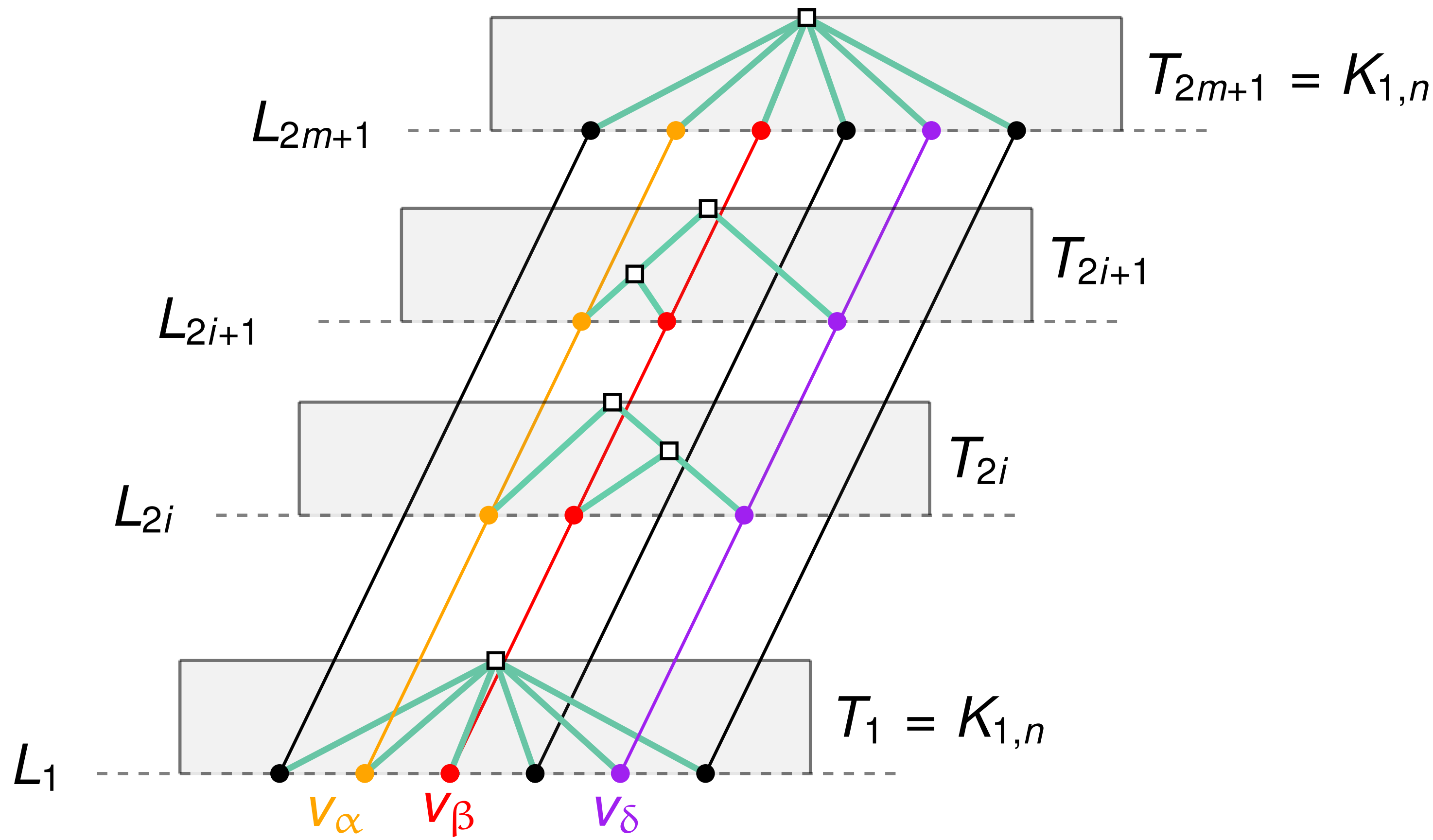

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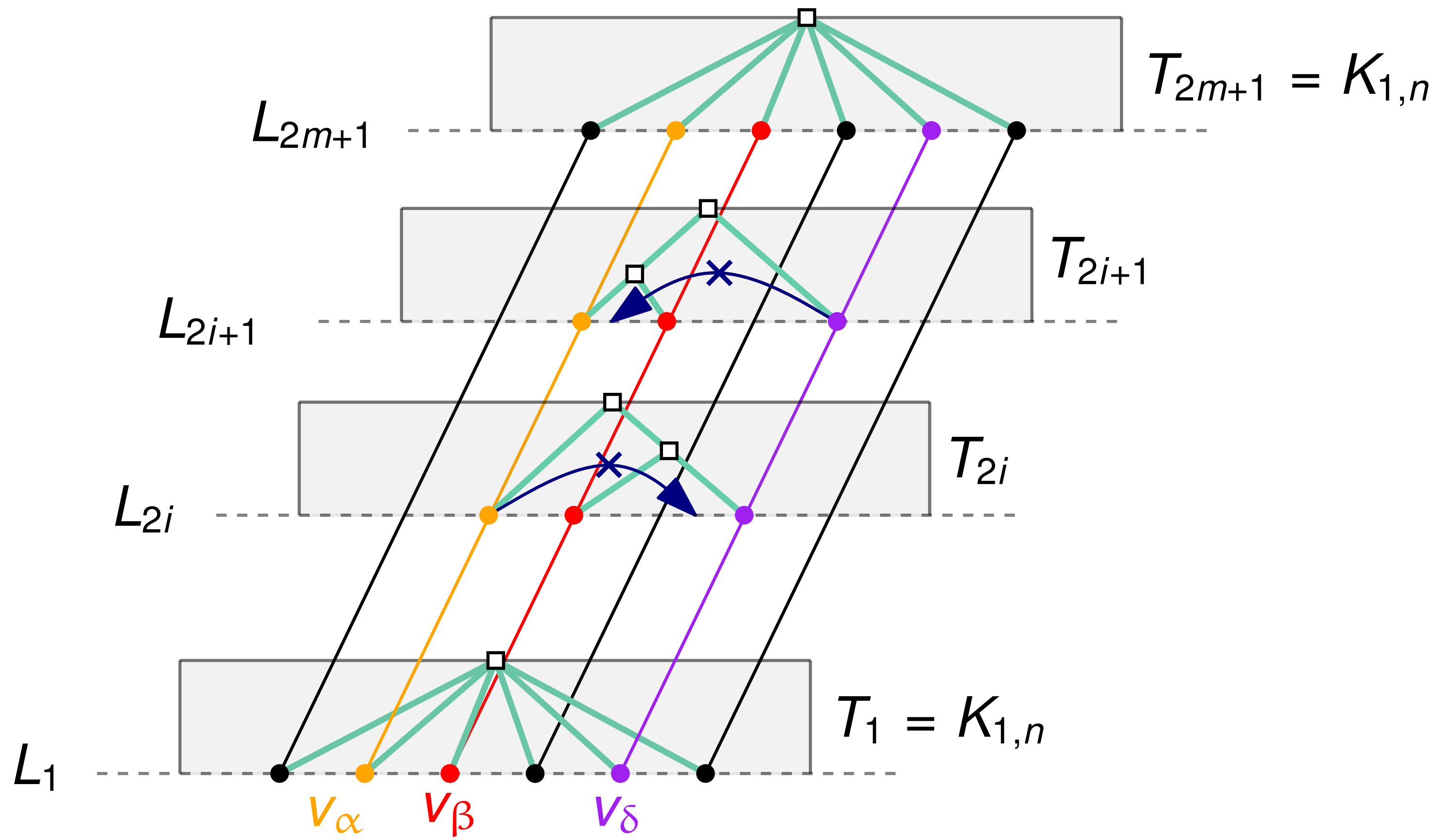


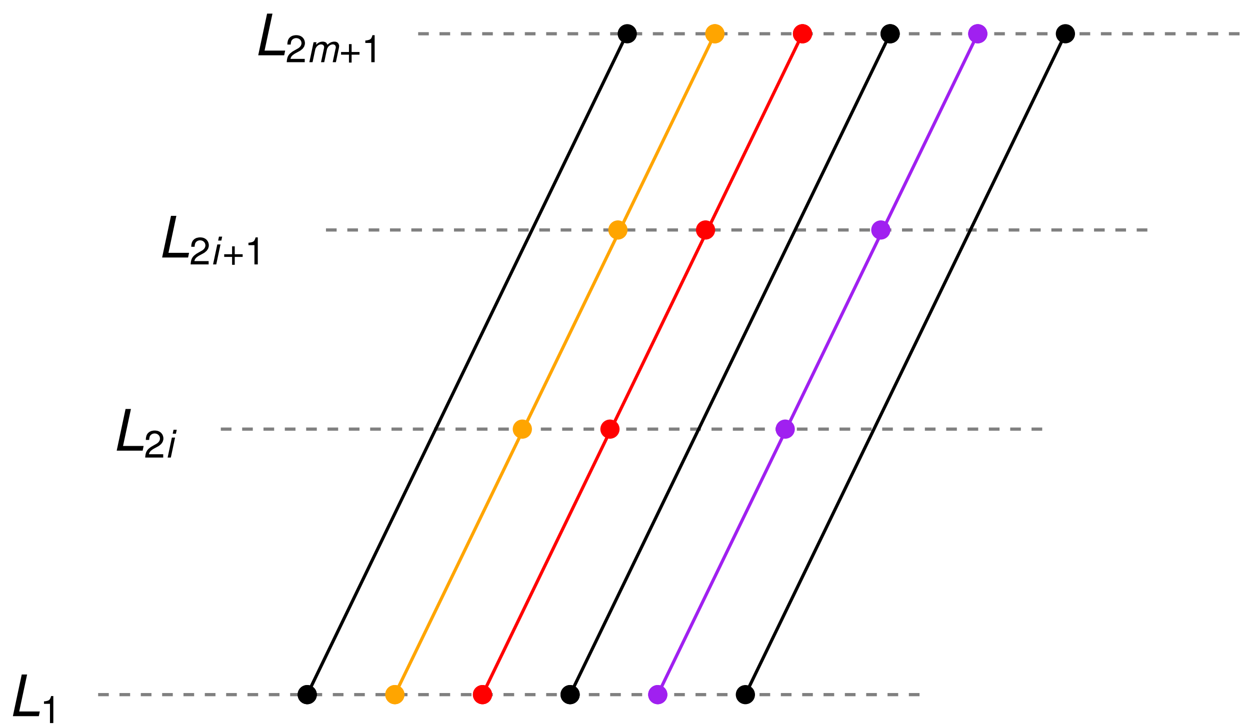
CL-PLANARITY is $\mathcal{NP}$-hard

BETWEENNESS

↓

T-LEVEL PLANARITY → $(V, E, \gamma, \mathcal{T})$ of CL-PLANARITY

- Graph (V, E) is a set of paths
- Hierarchy tree T is non-flat



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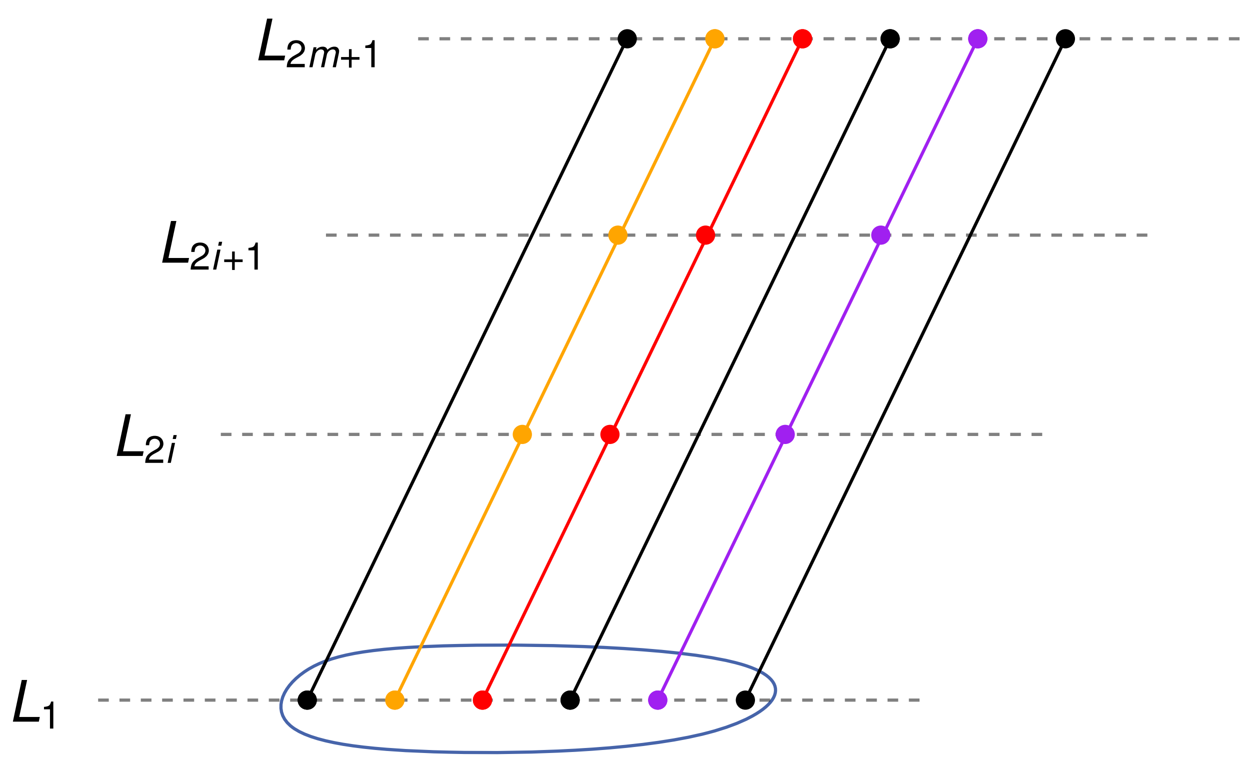

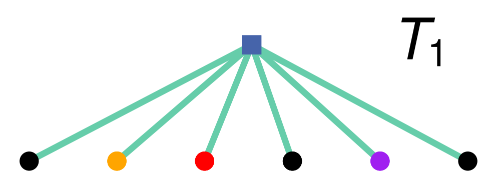

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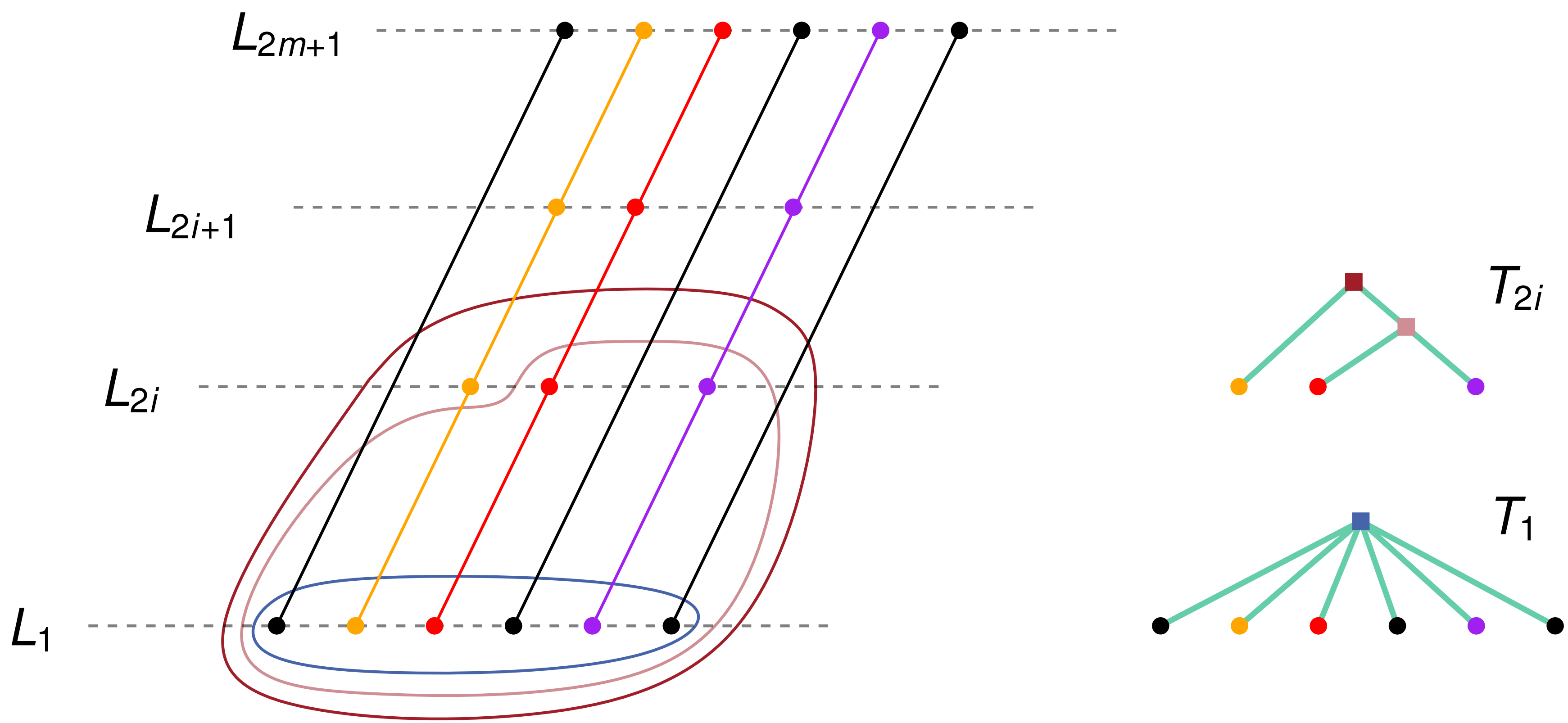

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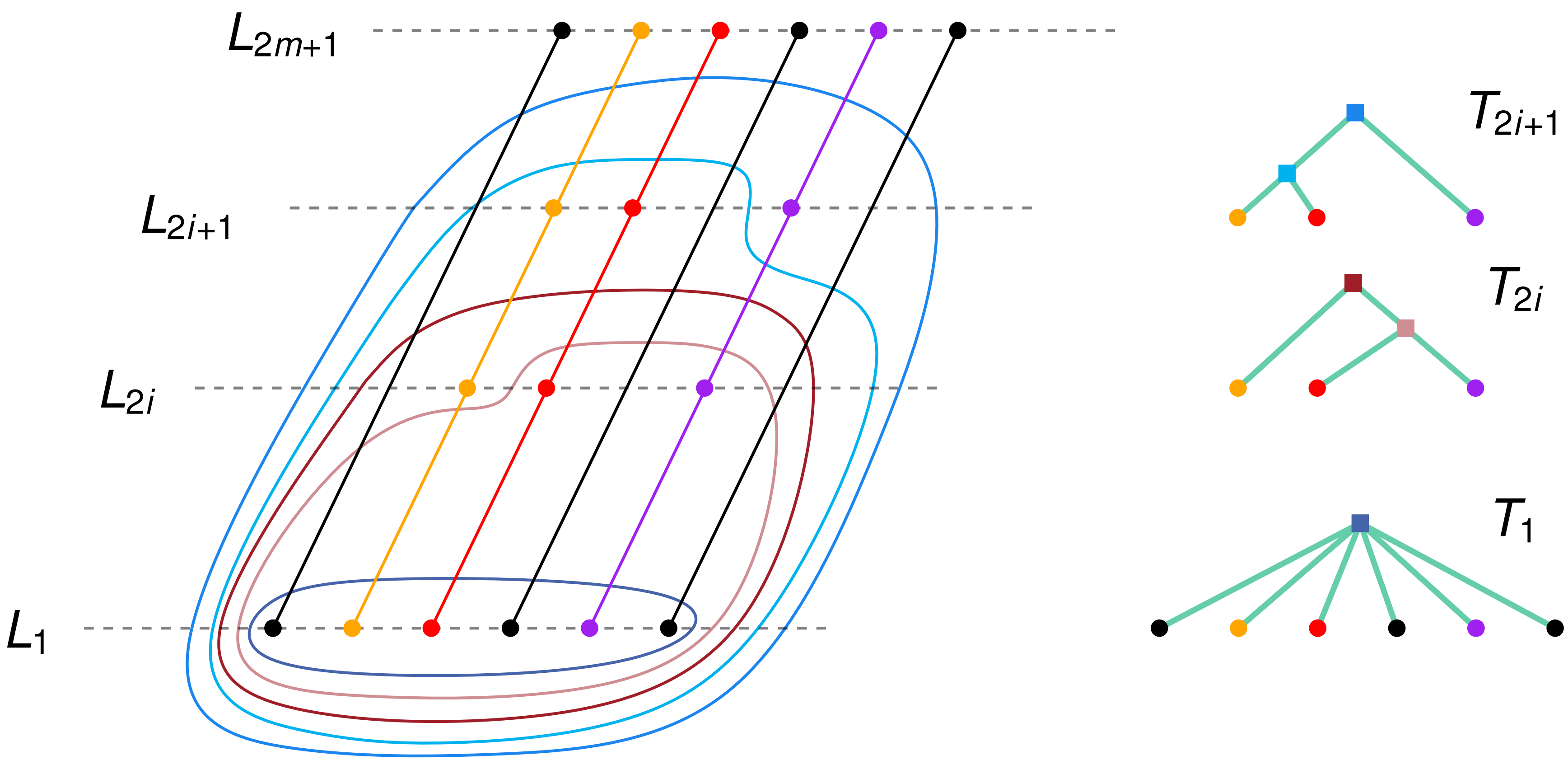

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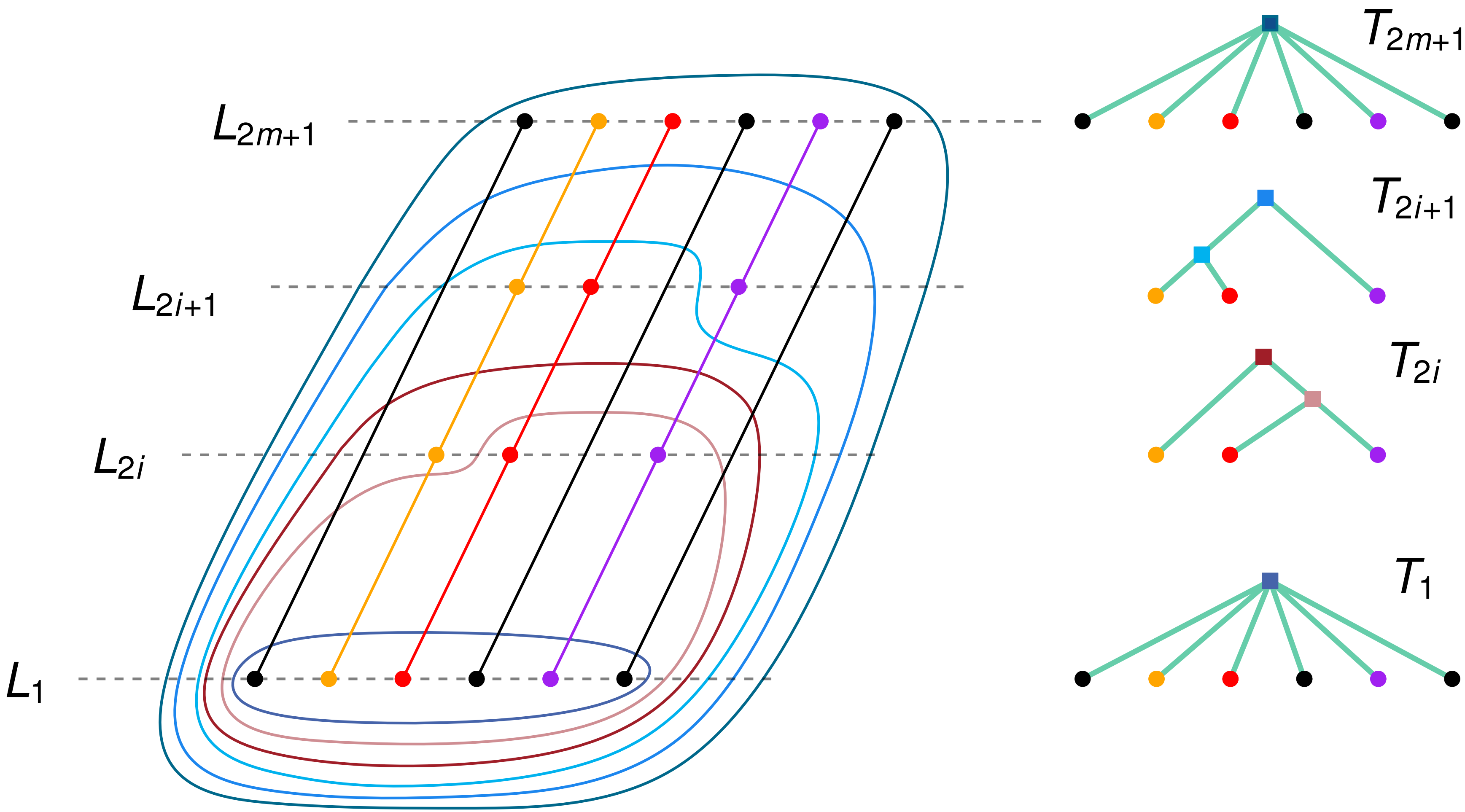

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BETWEENNESS

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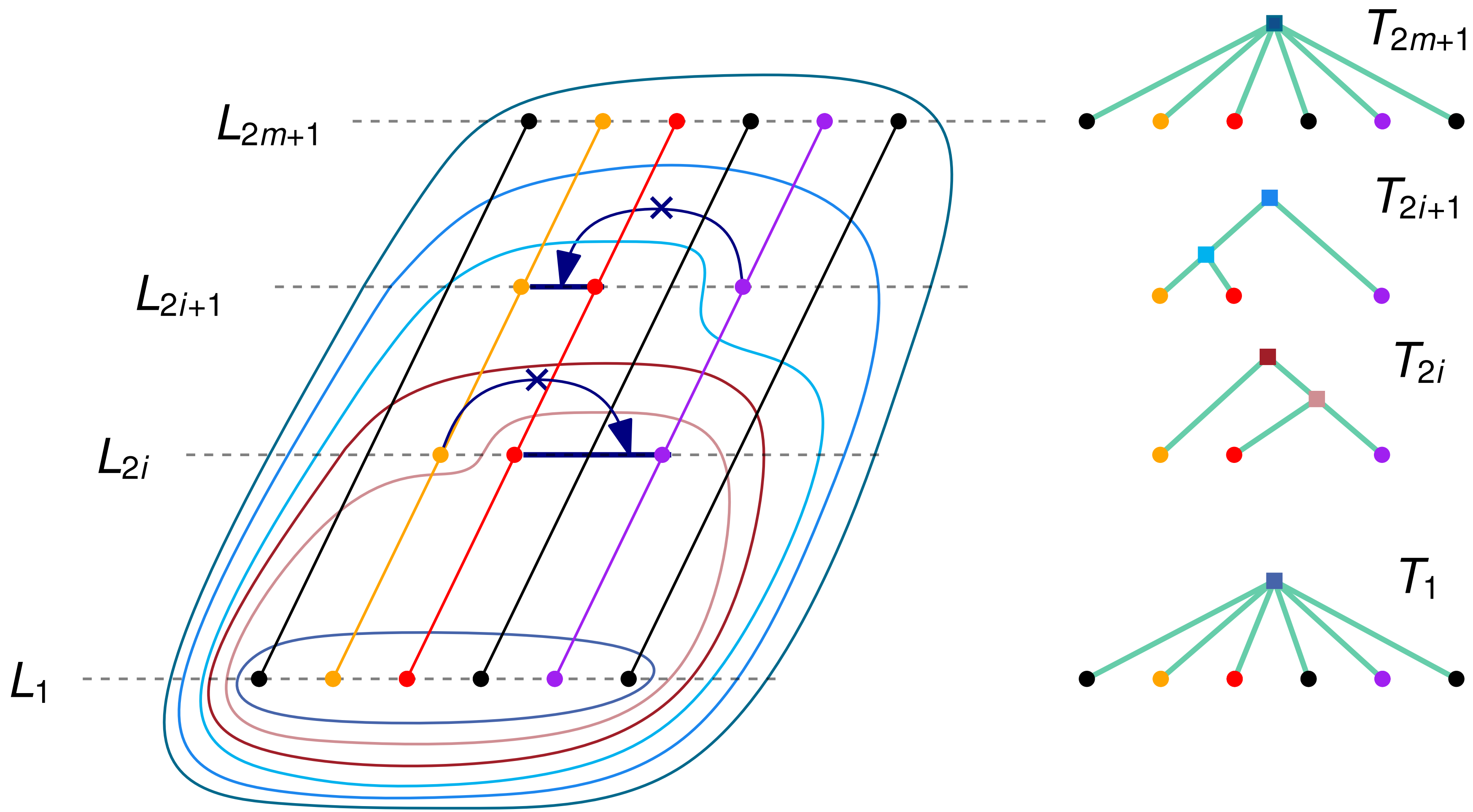

Complexity results : proper instances

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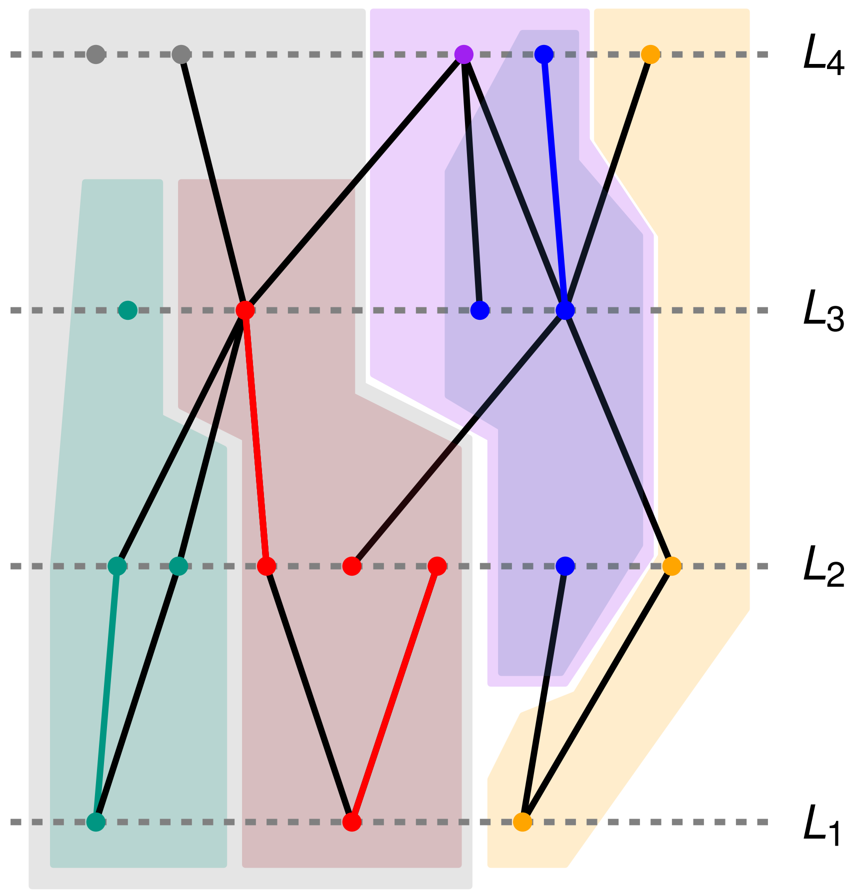
Clusters connectivity across levels



Level connectivity of a **proper cl-graph** $\begin{cases} \textbf{μ-level connected bw } L_i \textbf{ and } L_{i+1} \\ \textbf{μ-level connected} \\ \textbf{level-connected} \end{cases}$

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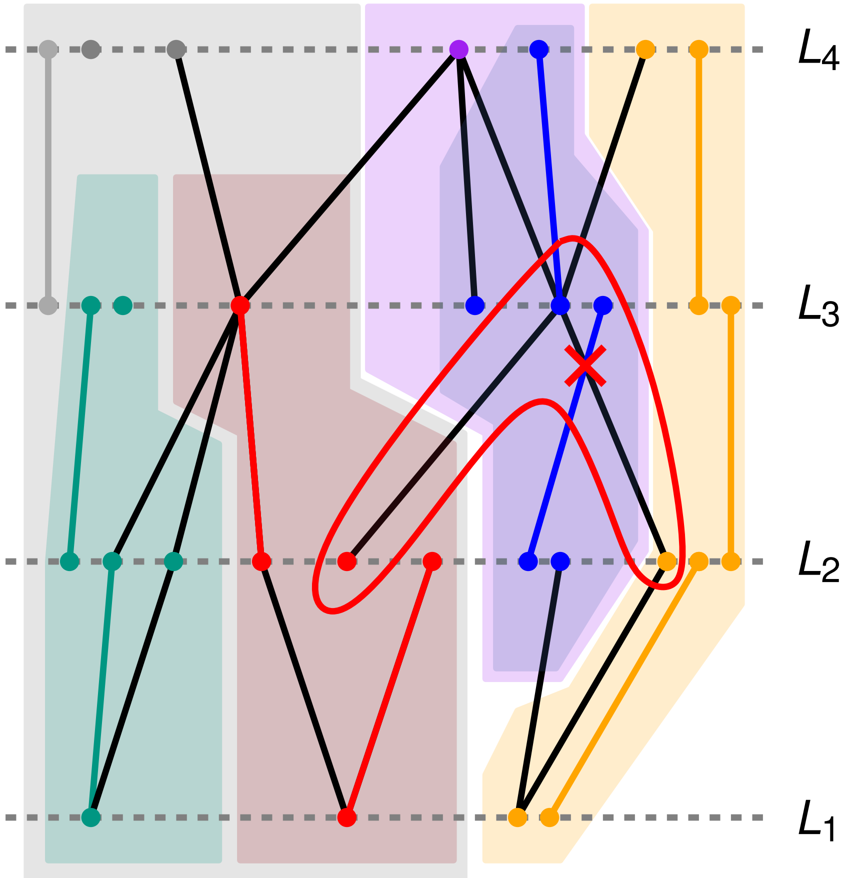

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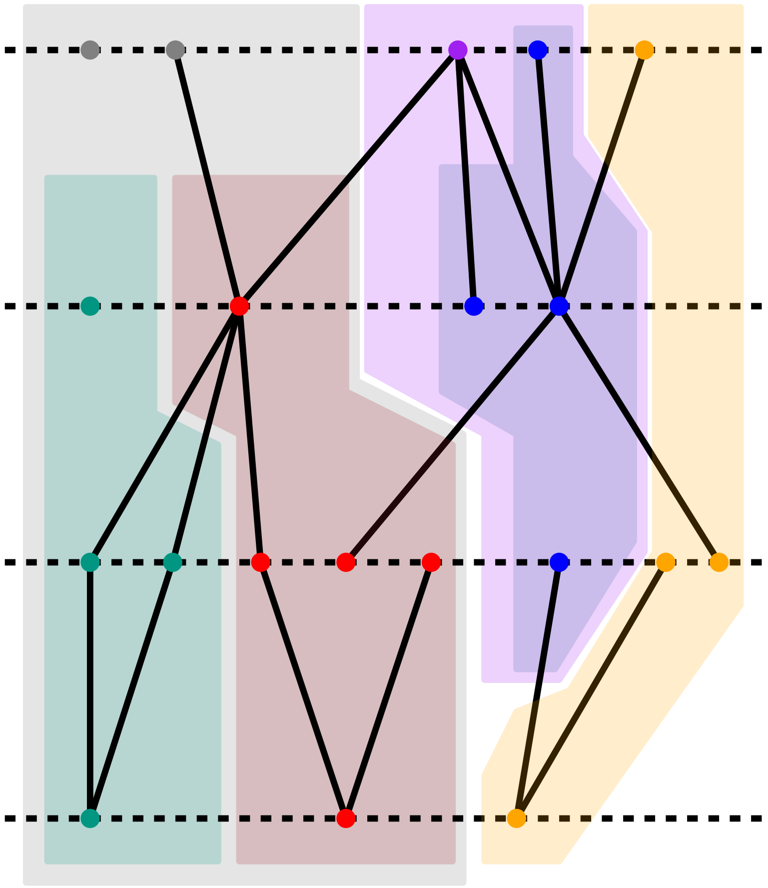
Level-connectivity doesn't matter!

Lemma 1

Let (V, E, γ, T) be a **proper** instance of **CL-Planarity**. An equivalent **level-connected** instance $(V^*, E^*, \gamma^*, T^*)$ of **CL-Planarity** of size $O(|V|^2)$ can be constructed in $O(|V|^2)$ time.



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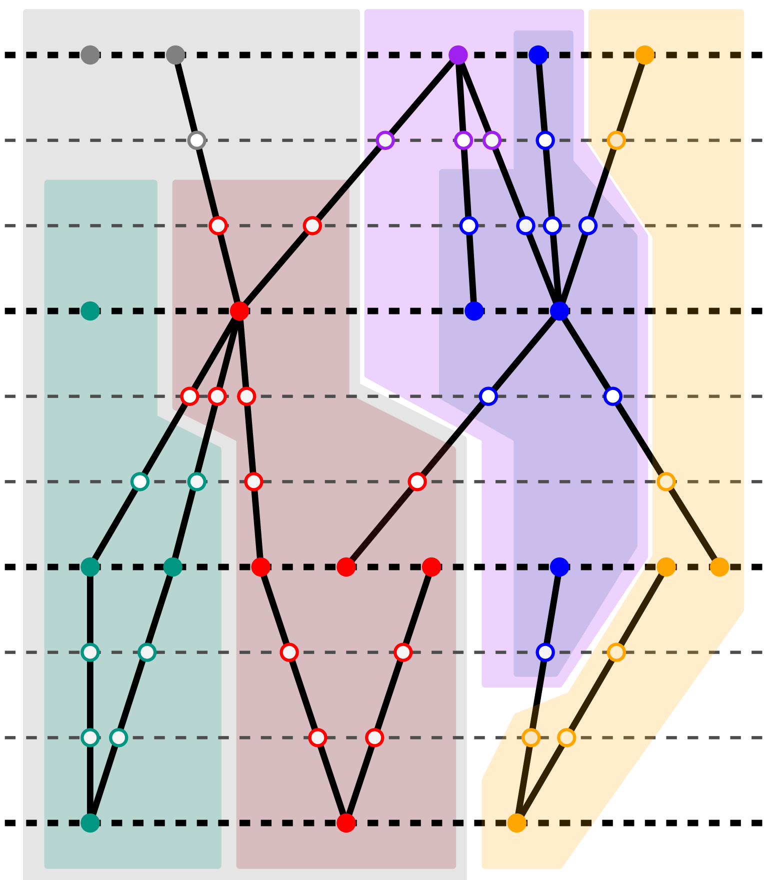
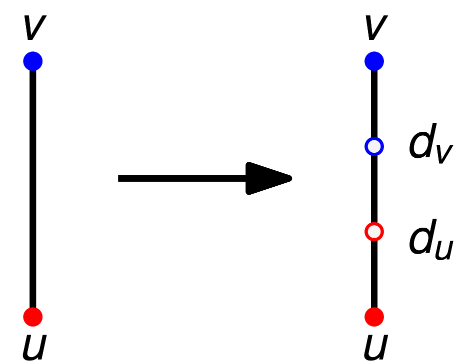


STEP 1



$v \in \mu_2 \rightarrow d_v \in \mu_2$

$u \in \mu_1 \rightarrow d_u \in \mu_1$

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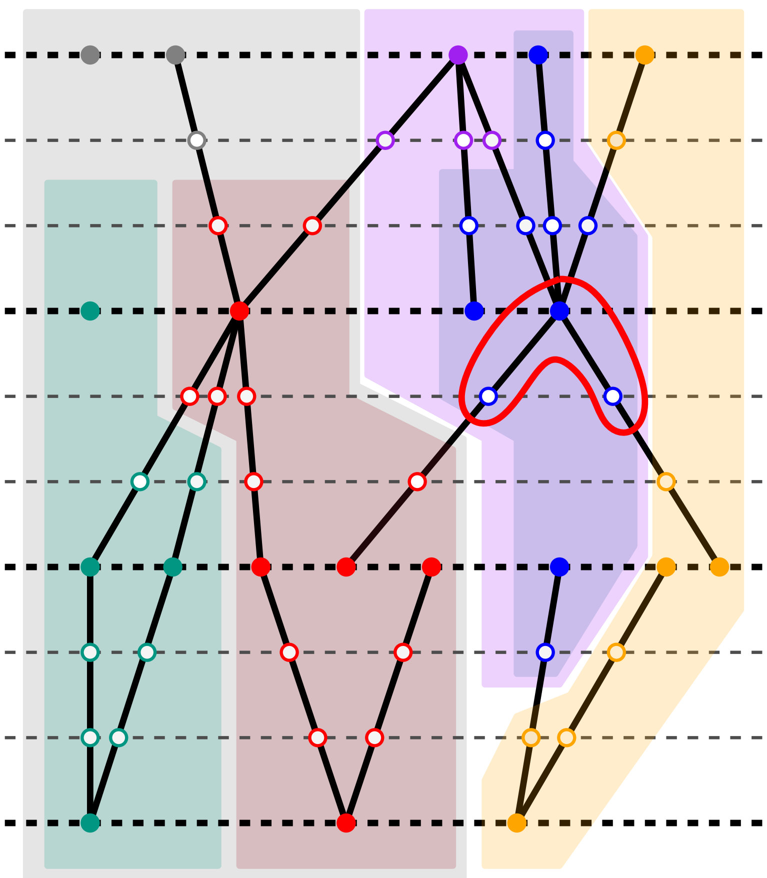

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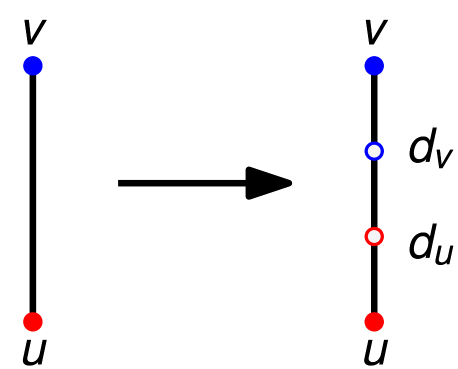

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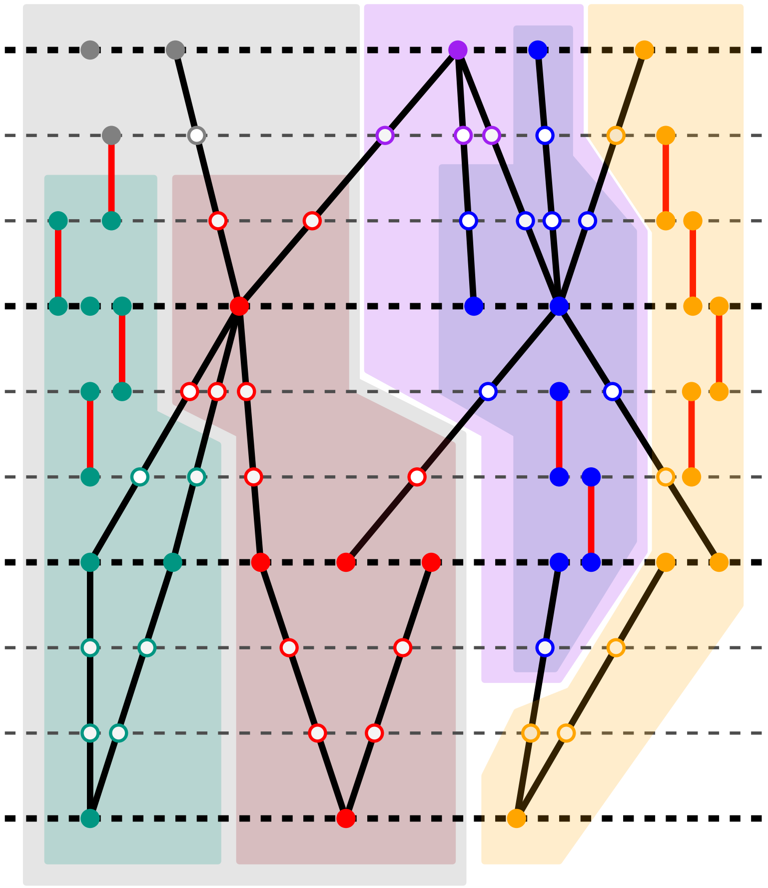


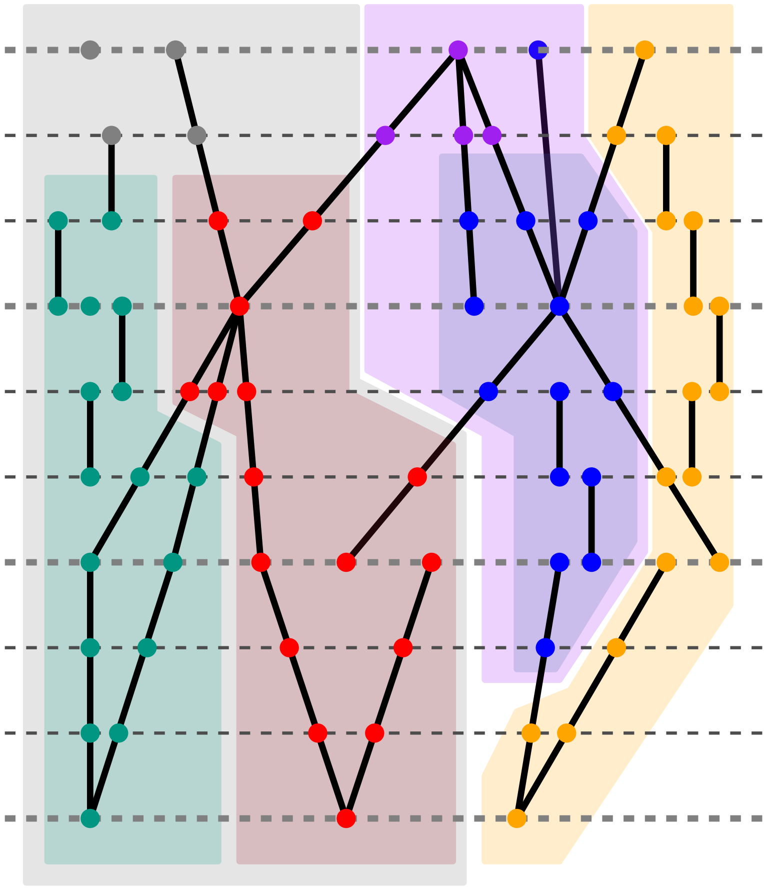
STEP 2

- $\mu \leftarrow$ bottom-up traversal of T
- for $i = min(\mu), \ldots, max(\mu)$, if (V, E, γ, T) is **not μ-level-connected bw $\mathbf{L_i}$ and $\mathbf{L_{i+1}}$**
  then
  "add a dummy edge bw L_i and L_{i+1}"

From CL-PLANARITY to T-LEVEL PLANARITY

Lemma 2

Let (V, E, γ, T) be a **(proper) level-connected** instance of **CL-Planarity**. An equivalent instance **proper** $(V, E, \gamma, \mathcal{T})$ of **T-Level Planarity** of size $O(|V|)$ can be constructed in $O(|V|)$ time.



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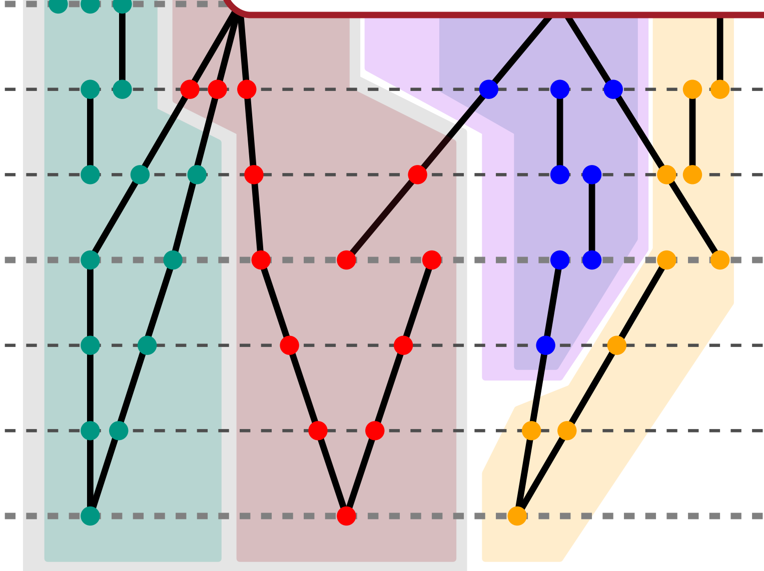
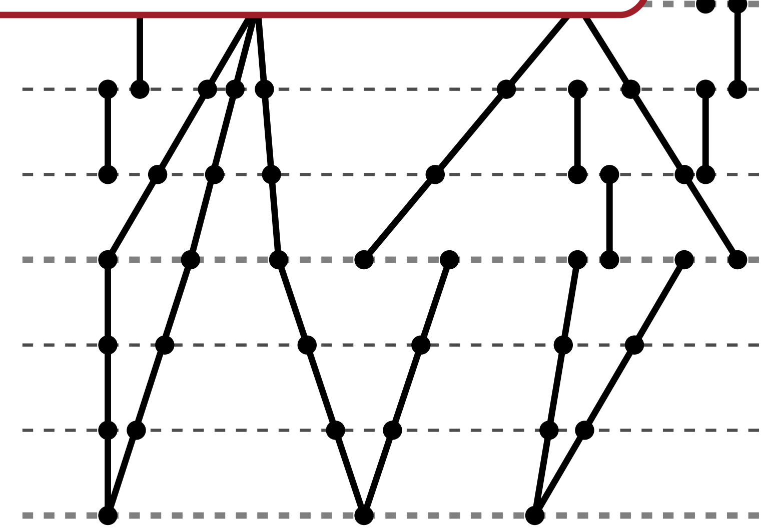
Procedure:

- The underlying level graph is (V, E, γ)
- ...





From CL-Planarity to T-Level Planarity

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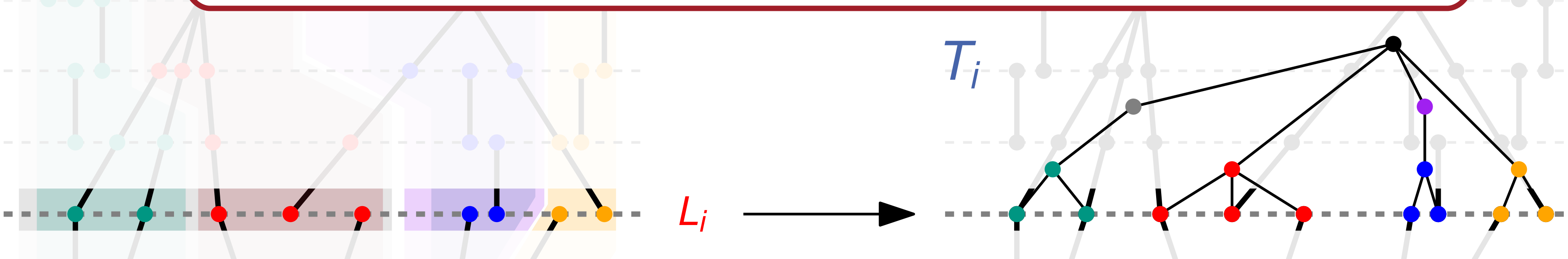
Procedure:

- The underlying level graph is (V, E, γ)
- for $i = 1, \ldots, k$, $T_i \in \mathcal{T}$ is the subtree of the cluster hierarchy T whose leaves belong to L_i

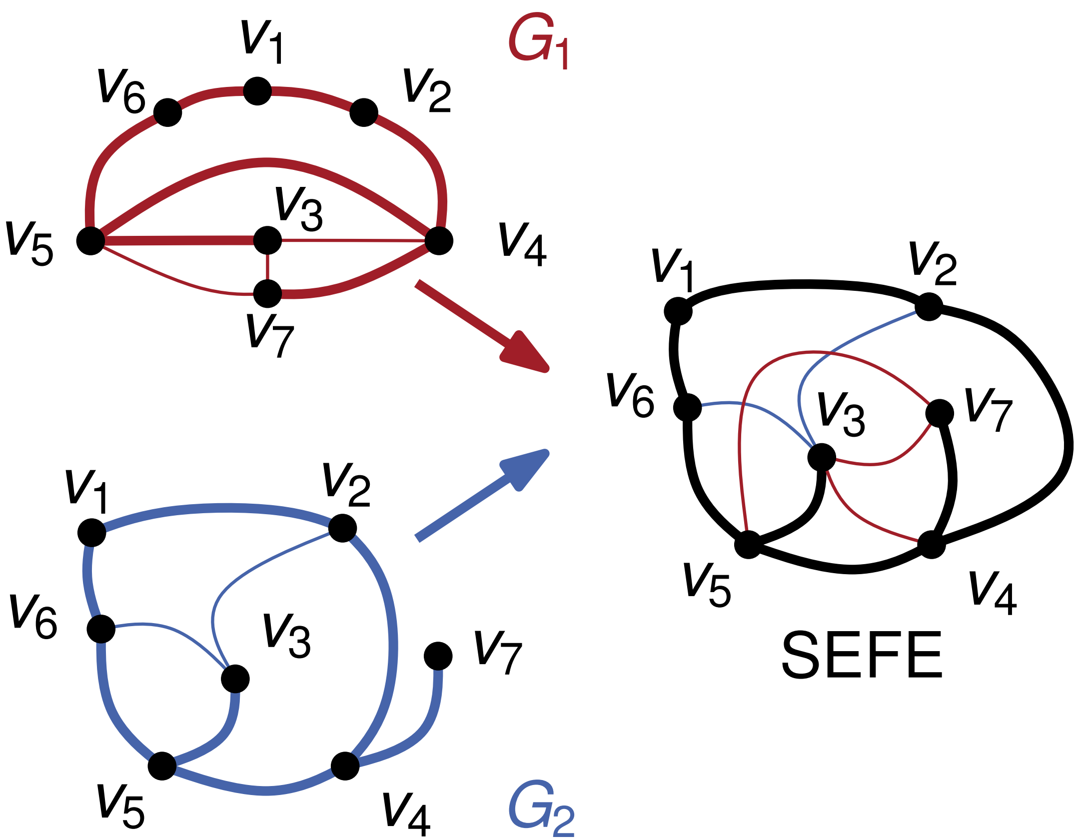


- T_i forces the vertices of each cluster to be consecutive along L_i
- **level-connectedness** and **level-planarity** impose that vertices of any two clusters have the same **relative order** in all levels

Simultaneous Embedding with FE (SEFE$_k$)

Problem Definition

- **input**: k planar graphs $G_1 = (V, E_1), G_2 = (V, E_2), \ldots, G_k = (V, E_k)$
- **question:** is there a SEFE of such graphs?



- k planar drawings $\Gamma_1, \Gamma_2, \ldots, \Gamma_k$

- for any $v \in V$, $\Gamma_i(v) = \Gamma_j(v)$
- for any $e \in E_i \cap E_j$, $\Gamma_i(e) = \Gamma_j(e)$

From T-Level Planarity to SEFE$_2$

Theorem 6.9, Corollary 6.10 [Schaefer - GD'12]

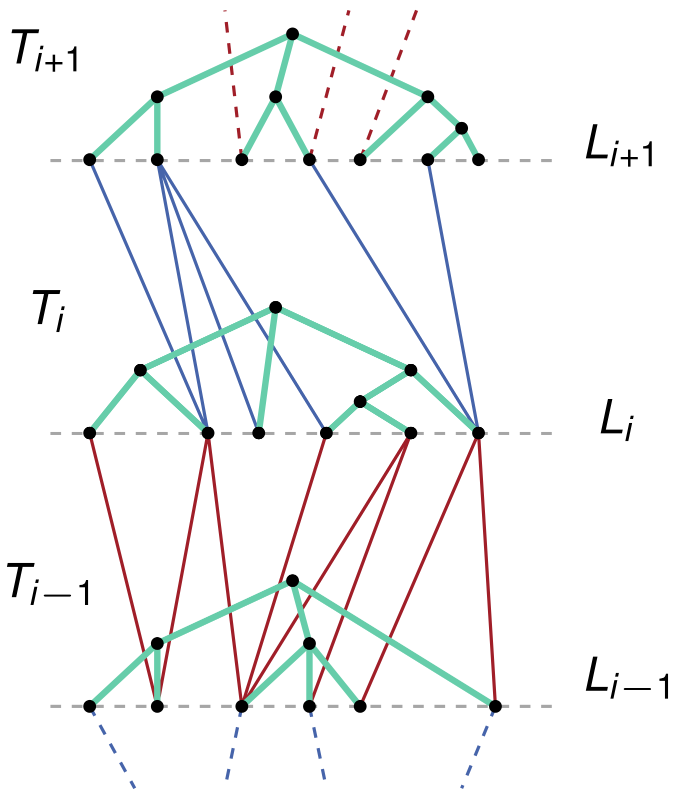
Given a **proper** instance $(V, E, \gamma, \mathcal{T})$ of T-LEVEL PLANARITY, deciding T-LEVEL PLANARITY reduces to the **SEFE$_2$** problem

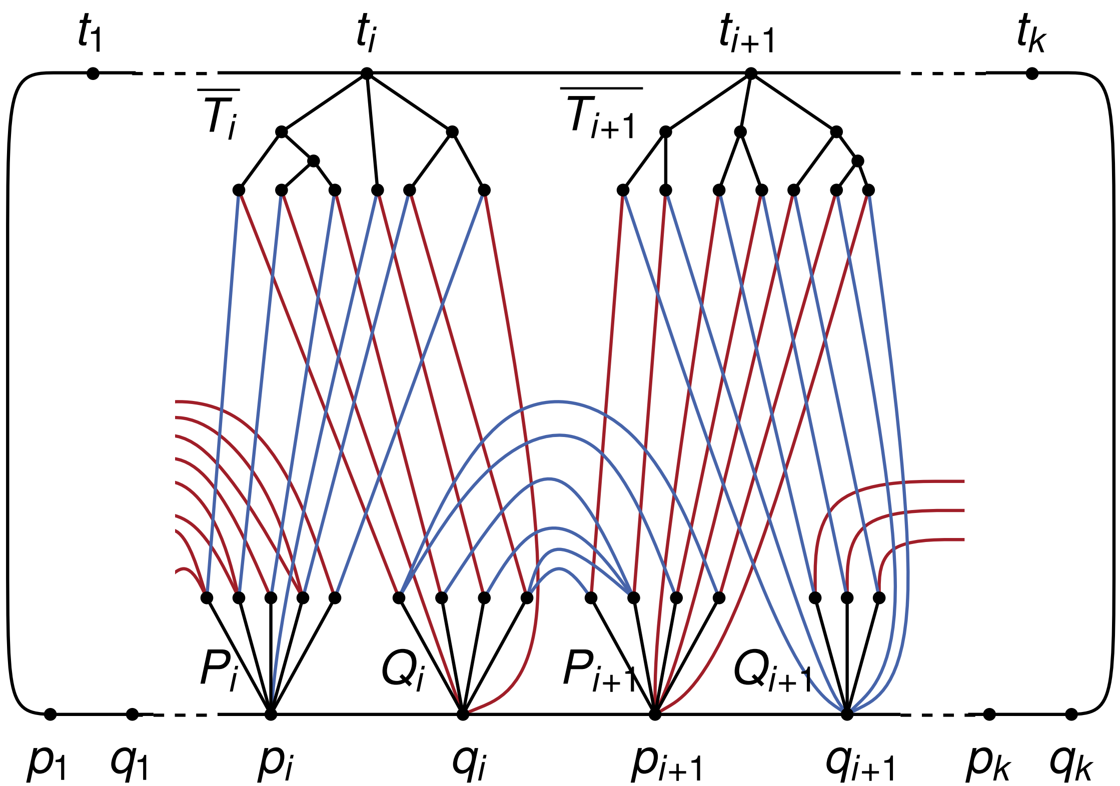
From T-Level Planarity to SEFE$_2$

Theorem

Given a **proper** instance $(V, E, \gamma, \mathcal{T})$ of T-Level Planarity, deciding T-Level Planarity reduces to the **SEFE$_2$** problem, where:

1. G_1 and G_2 are **2-connected**
2. $G_\cap$ is a **connected**





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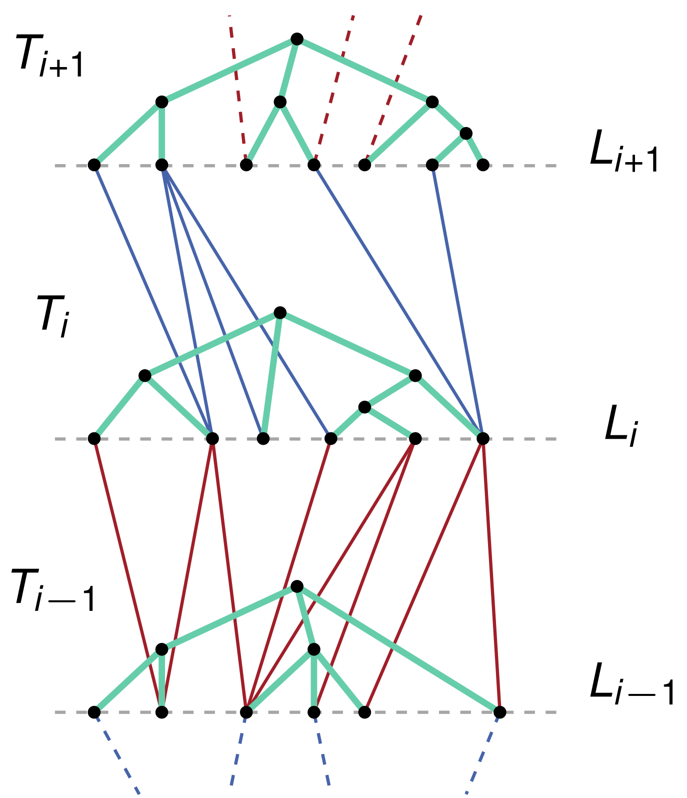
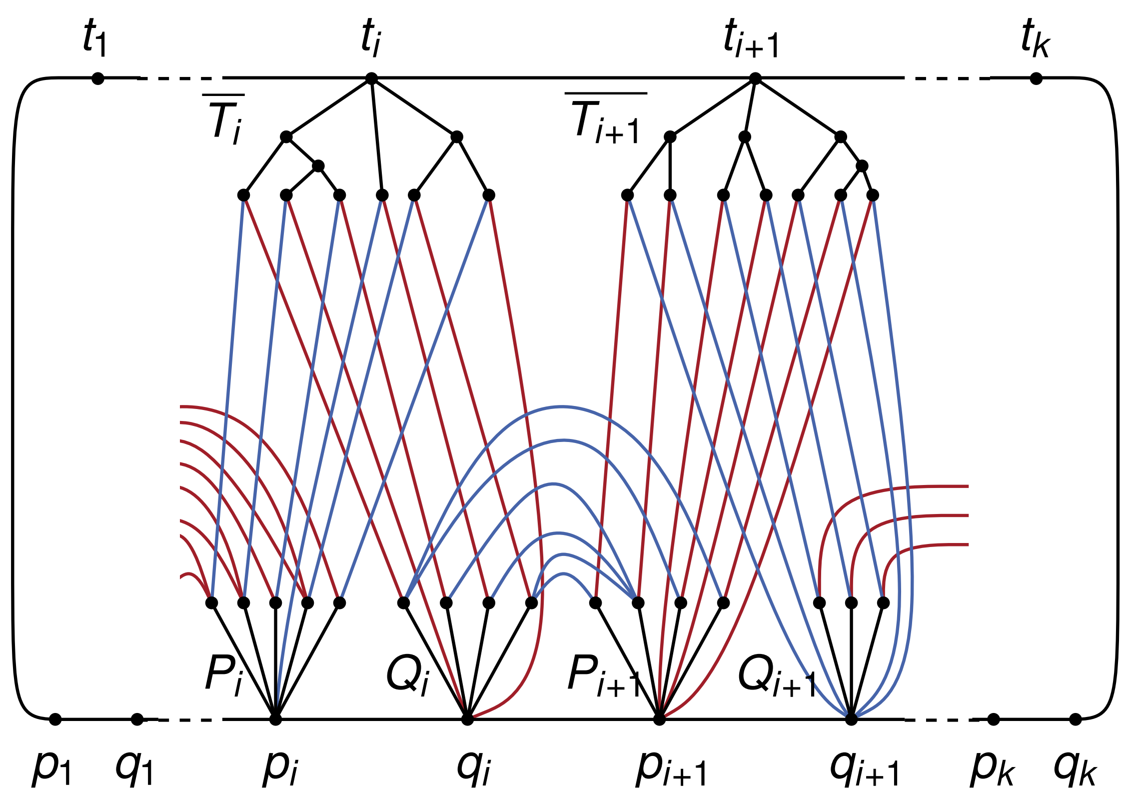
Th. 4.7 [Bläsius and Rutter - SODA'13]

$O(n^2)$-time testing algorithm

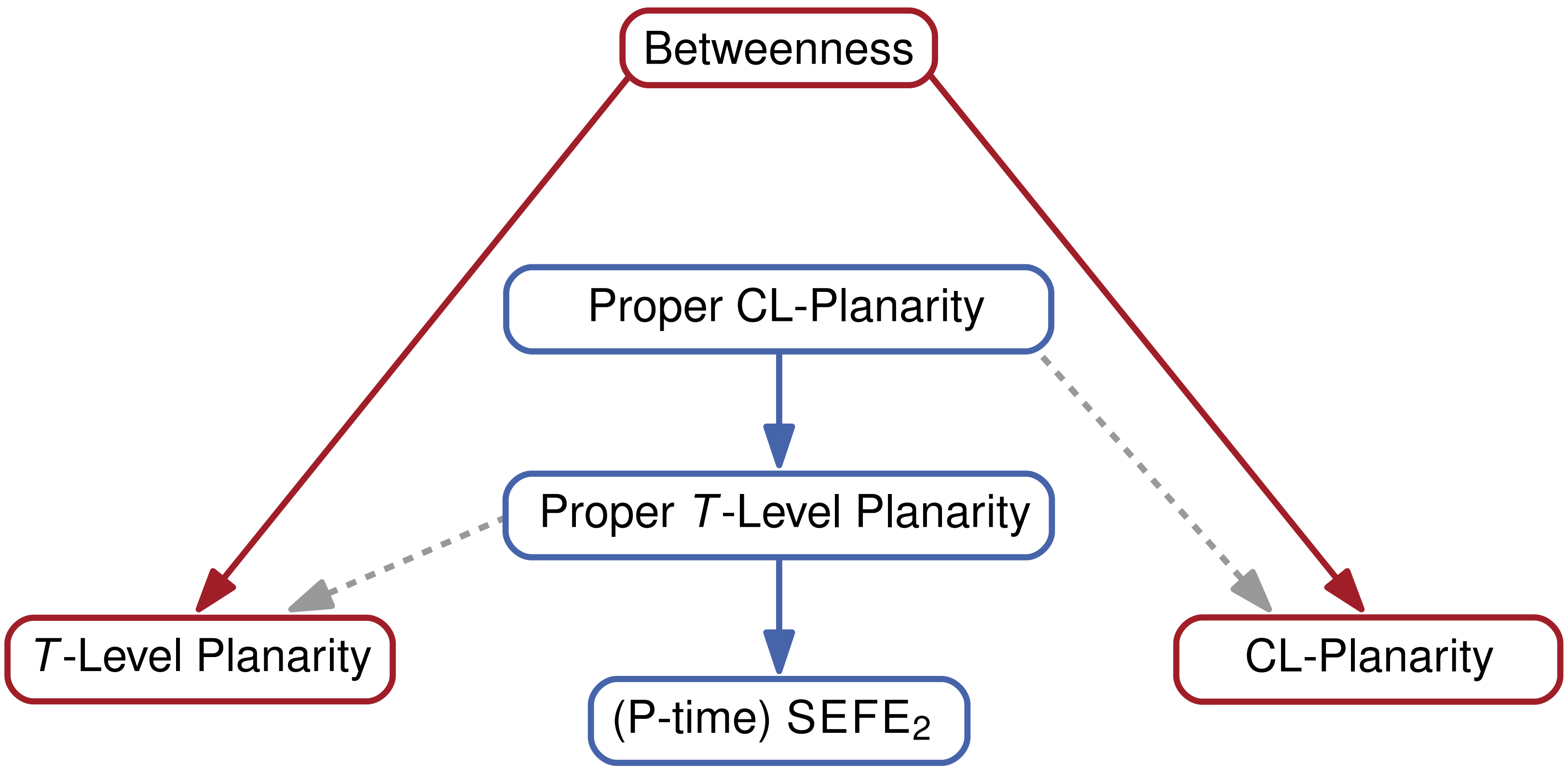




Main Results

Clustered-Level Planarity and T-Level Planarity are:

- $\mathcal{NP}$-Complete for **non-proper** instances
- polynomial-time solvable for **proper** instances
- ...



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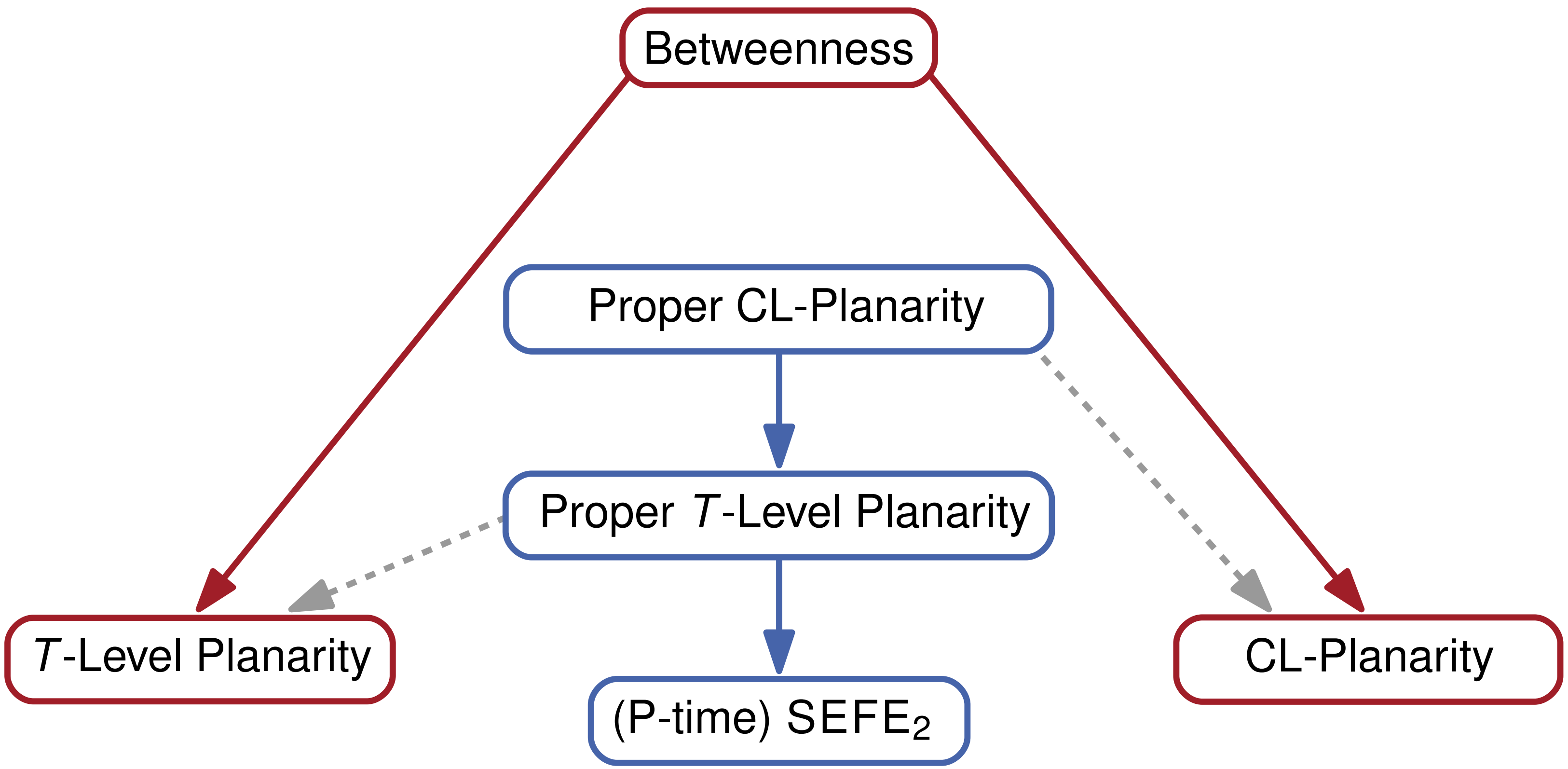
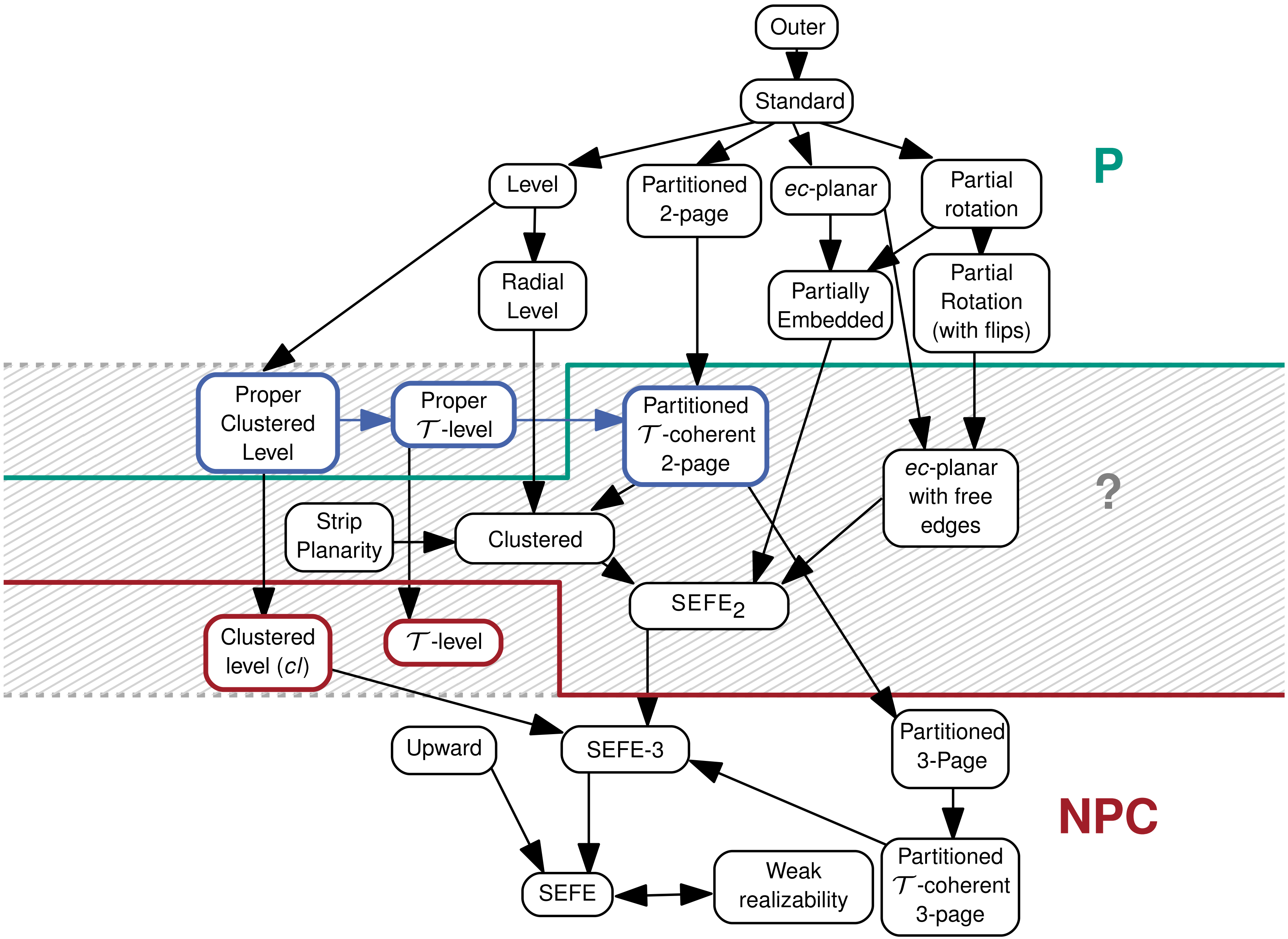
- polynomial-time solvable for **proper** instances
- **Open question [Schaefer, GD'12]**: CL-PLANARITY $\propto$ SEFE$_2$?



Reducibility between Planarity Variants



Open Problems

T-LEVEL PLANARITY and CLUSTERED-LEVEL PLANARITY

1. improving the complexity bounds for proper instances
   - Recall that, a linear-time testing algorithm for T-LEVEL PLANARITY would also imply a quadratic-time testing algorithm for CL-PLANARITY
2. Is CL-PLANARITY still $\mathcal{NP}$-hard if the cluster hierarchy is flat?

C-PLANARITY

1. Is it possible to use similar techniques to tackle the problem of determining the complexity of C-PLANARITY?
   - Recall that, in the CLUSTERED-LEVEL PLANARITY problem none of the C-PLANARITY constraints is dropped

Coming soon on Springer...

Thank you for your attention!

Coming soon on Springer...