Morphing Schnyder drawings of planar triangulations

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Outline

Introduction

Morphs

Given two drawings Γ and Γ' of a graph G, a *morph* between Γ and Γ' is a continuous family of drawings of G, $\{\Gamma^t\}_{t\in[0,1]}$, such that $\Gamma^0 = \Gamma$ and $\Gamma^1 = \Gamma'$.

Planar morphs

Let G be a graph and let $M = \{\Gamma^t\}_{t \in [0,1]}$ be a morph between the drawings Γ and Γ' of G.

Planar morph

We say M is *planar* if Γ^t is a planar drawing for all $t \in [0, 1]$.

Linear morphs

Let G be a graph and let $M = \{\Gamma^t\}_{t\in[0,1]}$ be a morph between the drawings Γ and Γ' of G.

Linear morph

We call M a *linear* morph, denoted $\langle \Gamma^0, \Gamma^1 \rangle$, if each vertex moves from its position in Γ^0 to its position in Γ^1 along a line segment and at constant speed.

The morphing problem

Let T be a planar triangulation and let f be a face of T. Consider two planar drawings Γ and Γ' such that f is the unbounded face in both drawings.

Morphing problems

Does there exist a planar morph from Γ to Γ'?

Cairns (1944): Yes. Can be implemented with $O(2^n)$ linear morphing steps.

The morphing problem

Let T be a planar triangulation and let f be a face of T. Consider two planar drawings Γ and Γ' such that f is the unbounded face in both drawings.

Morphing problems

Is there an efficient algorithm to morph between any Γ and Γ'?

Floater & Gotsman (1999): Yes. Based on Tutte's method for drawing a graph. Trajectories followed by vertices may be complex.

The morphing problem

Let T be a planar triangulation and let f be a face of T. Consider two planar drawings Γ and Γ' such that f is the unbounded face in both drawings.

Morphing problems

Is there an efficient algorithm that uses a polynomial number of linear morphing steps?

Alamdari et al. (2013): Yes. We can morph between any two drawings in $O(n^2)$ linear morphing steps. Vertices may become arbitrarily close during the morph.

The morphing problem

Let T be a planar triangulation and let f be a face of T. Consider two planar drawings Γ and Γ' such that f is the unbounded face in both drawings.

Morphing problems

Is there an efficient algorithm that uses a polynomial number of linear morphing steps?

Angelini et al. (2014): Yes. We can morph between any two drawings in $O(n)$ linear morphing steps. Vertices may become arbitrarily close during the morph.

Our result

Morphing between Schnyder drawings

We show that it is possible to morph between any two *Schnyder drawings* using $O(n^2)$ linear morphing steps while remaining in an $O(n) \times O(n)$ grid.

Nano-course on Schnyder woods

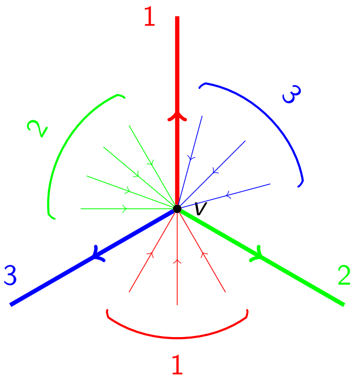
Schnyder woods

A *Schnyder wood* S of a planar triangulation T with respect to a face $f = a_1 a_2 a_3$ is an assignment of directions and colours 1, 2 and 3 to the interior edges of T such that the following two conditions hold.

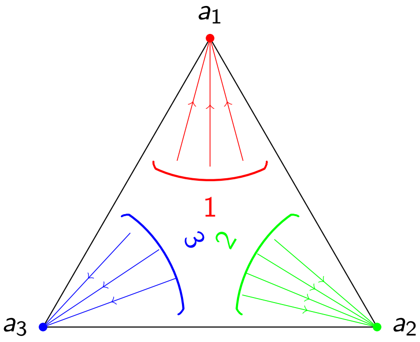
(D1) Each interior vertex v has outdegree 1 in colour i, $i = 1, 2, 3$. At v, the outgoing edge in colour $i - 1$, e_{i-1}, appears after the outgoing edge in colour $i + 1$, e_{i+1}, in clockwise order. All incoming edges in colour i appear in the clockwise sector between the edges e_{i+1} and e_{i-1}.

(D2) At the exterior vertex a_i, all the interior edges are incoming and of colour i.

Schnyder woods



(D1)

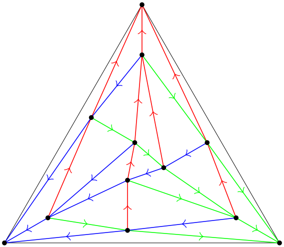


(D2)

Schnyder woods

Theorem (Schnyder 89)

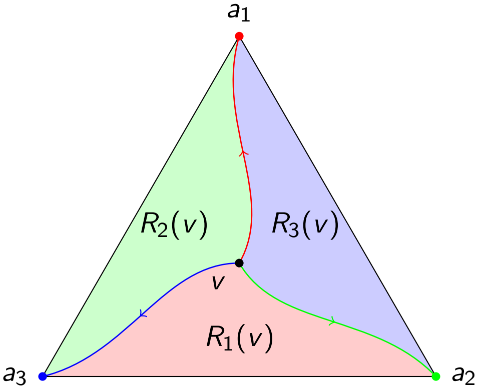
Every planar triangulation has a Schnyder wood.



Planar drawings from Schnyder woods

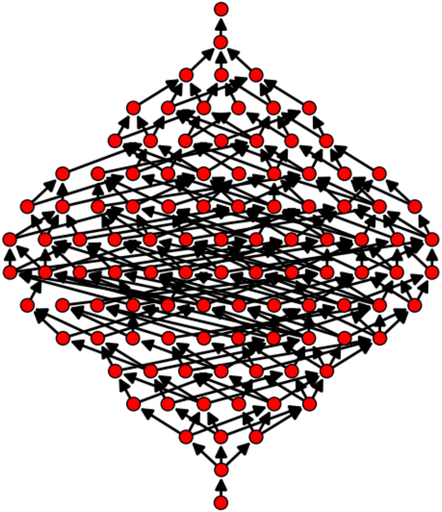
A Schnyder wood of a planar triangulation T induces a partition of the set of interior faces of T into 3 regions.

A planar straightline drawing of T in an $O(n) \times O(n)$ grid may be obtained by mapping each vertex v to the point $(|R_1(v)|, |R_2(v)|, |R_3(v)|)$.



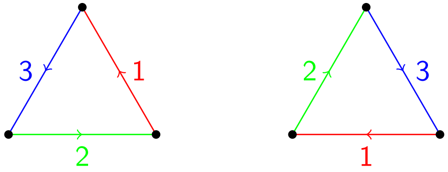
The Schnyder wood lattice

In general, the number of Schnyder woods of a fixed planar triangulation may be exponential (Felsner & Zickfeld, 2007).



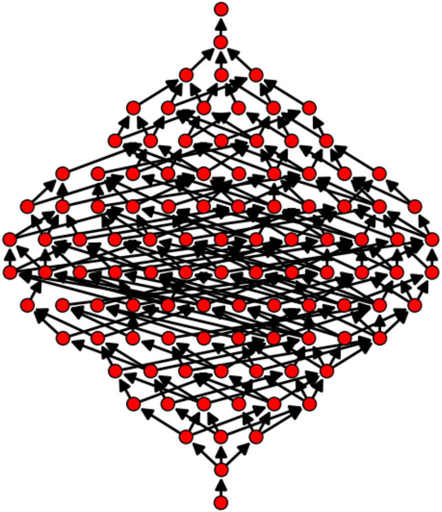
The Schnyder wood lattice

It is known that the set of Schnyder woods has the structure of a distributive lattice and that the basic operation to traverse such lattice is by reversing cyclically oriented triangles and “cyclically” recolouring any edges bounded by the cycle (if any). We call such an operation a *flip* of a triangle (Brehm 2000,Felsner 2004,Ossona de Mendez 1994).



The Schnyder wood lattice

It can be shown from Brehm's results that the maximum distance in the lattice between two Schnyder woods is $O(n^2)$.



Our result

Morphing between Schnyder drawings

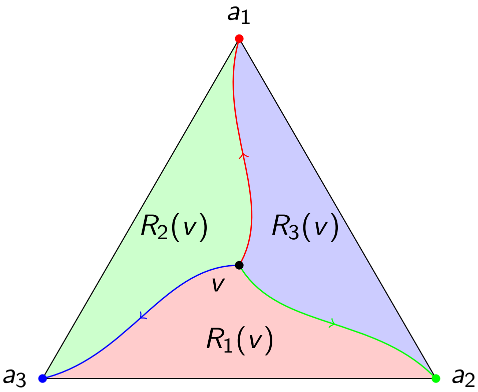
We show that it is possible to morph between any two *Schnyder drawings* using $O(n^2)$ linear morphing steps while remaining in an $O(n) \times O(n)$ grid.

Morphing through the set of Schnyder drawings

Morphs by weight shifts

(Dhandapani 2008)

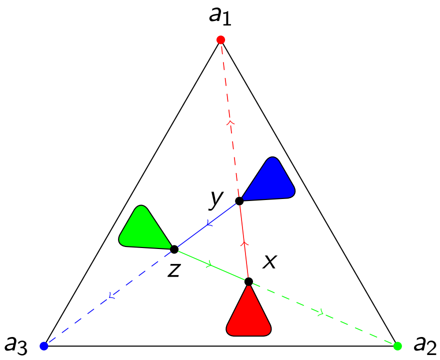
A *weight distribution* is an assignment of positive weights to the set of interior faces such that the total weight distributed is $2n - 5$. Given two drawings Γ and Γ' given by a Schnyder wood S and weight distributions $\mathbf{w}$ and $\mathbf{w}'$, the morph given by considering S and $\mathbf{w}^t := (1-t)\mathbf{w} + \mathbf{w}'$ defines the planar linear morph $\langle \Gamma, \Gamma' \rangle$.

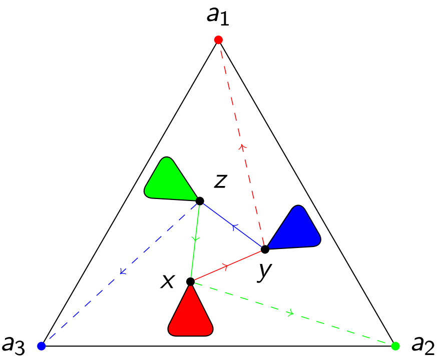


Morphs from flips on faces

Theorem

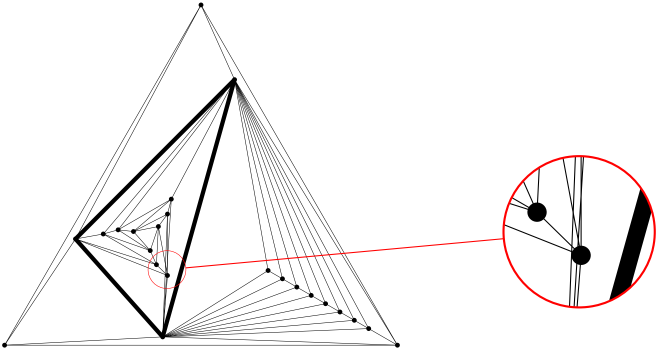
Let T be a planar triangulation, S be a Schnyder wood of T, f be a flippable face in S and $\mathbf{w}$ be a weight distribution. If $\Gamma = D[S, \mathbf{w}]$ and $\Gamma' = D[S^f, \mathbf{w}]$, then $\langle \Gamma, \Gamma' \rangle$ is a planar linear morph.





Non 4-connected planar triangulations

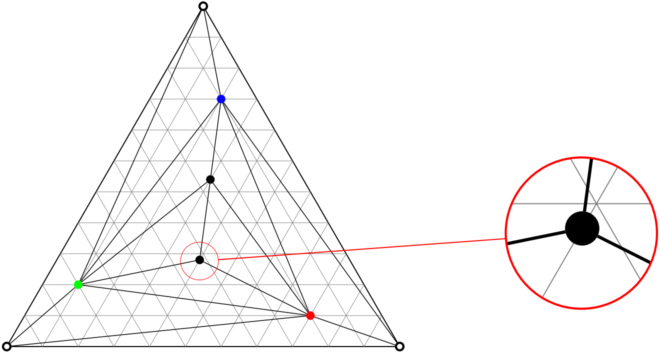
The previous theorem does not hold in general for flippable separating triangles.



An alternate approach

A possible approach would be to decompose the planar triangulation into 4-connected blocks and perform flips on each block.

This approach has the disadvantage that the size of the grid may increase to $O(n^k) \times O(n^k)$, where k is the *depth* of the block decomposition.

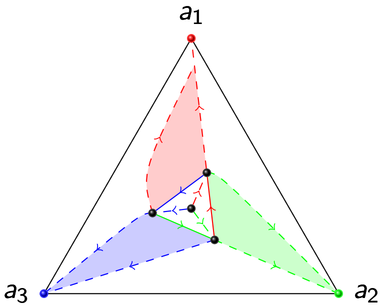


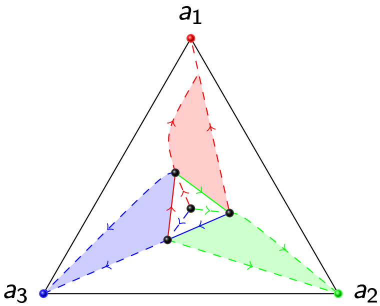
Morphs from flips on separating triangles

Theorem

Let T be a planar triangulation, S be a Schnyder wood of T, f be a flippable separating triangle in S and $\mathbf{w}$ be a weight distribution. If $\Gamma_1 := D[S, \mathbf{w}]$ and $\Gamma_4 := D[S^f, \mathbf{w}]$, then there exists a weight distribution $\mathbf{w}'$ such that $\langle \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4 \rangle$ is a planar morph, where $\Gamma_2 := D[S, \mathbf{w}']$ and $\Gamma_3 := D[S^f, \mathbf{w}']$.

The weight distribution $\mathbf{w}'$ can be chosen such that Γ_2 and Γ_3 are realized in an $O(n) \times O(n)$ grid.





Traversing the Schnyder lattice using morphs

Morphing via flipping triangles yields the following result.

Theorem

Let $\Gamma := D[S, \mathbf{w}]$ *and* $\Gamma' := D[S', \mathbf{w}']$ *be Schnyder drawings of a planar triangulation* T*. There exists a sequence of Schnyder drawings of* T $\Gamma_1, \ldots, \Gamma_k$*,* $k = O(n^2)$*, such that*

- *the morph* $\langle \Gamma, \Gamma_1, \ldots, \Gamma_k, \Gamma' \rangle$ *is planar,*
- *the drawing* Γ_i *is realized in an* $O(n) \times O(n)$ *grid.*

Future work

Future work

- Any drawing of a planar triangulation T can be obtained from any Schnyder wood of T and some weight assignment on the set of interior faces. However, there exist drawings that cannot be realized using only positive weights. In this case we can morph from such drawing to a Schnyder drawing in $O(n)$ linear morphing steps (but allowing vertices to be arbitrarily close to each other).
  Q: Can we morph in $O(n^2)$ linear morphing steps (while gradually improving the size of the grid) from an arbitrary drawing to a Schnyder drawing in an $O(n) \times O(n)$ grid?
- **Q:** Can the Schnyder morphs be generalized to the class of 3-connected planar graphs to yield morphs that preserve convexity of faces?

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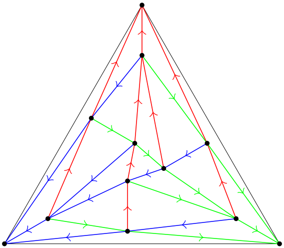

Thanks!