Height-preserving Transformations of Planar Graph Drawings

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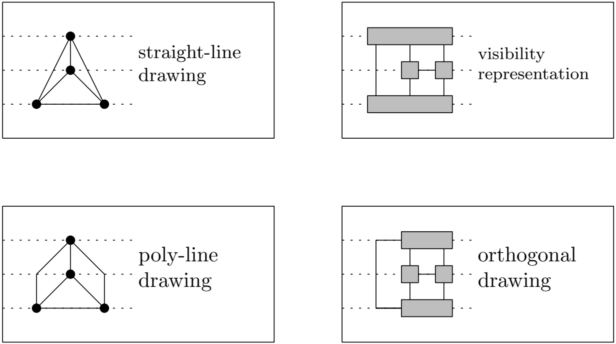
An extremely brief review...

- This talk: All graphs and all drawings are planar.
- Four drawing styles considered:

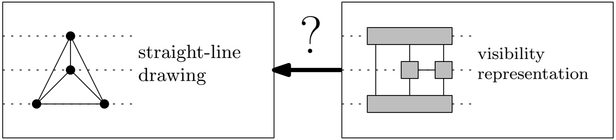


- Main objective: Small area, or at least small height.

Problem and Motivation

Problem

Given a drawing in style X, convert it into a drawing in style Y, without changing area/height (much).



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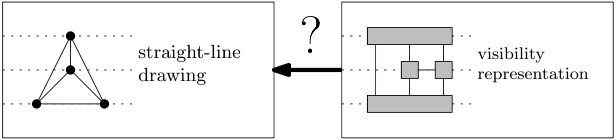

Motivation: How to draw outer-planar graphs?

- Visibility representation:
  $O(n \log n)$ area, $O(pw(G))$ height [B. 2002, 2012]

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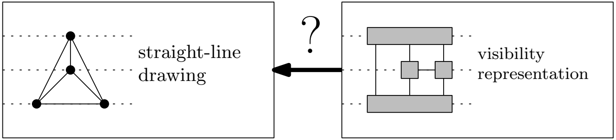

Motivation: How to draw outer-planar graphs?

- Visibility representation:
  $O(n \log n)$ area, $O(pw(G))$ height [B. 2002, 2012]
- Straight-line drawing???
  Area $O(n^{1.48})$ or $O(\Delta n \log n)$ [DiBattista & Frati 2005, 2007]
  Open: $O(n \log n)$ area? $O(pw(G))$ height?

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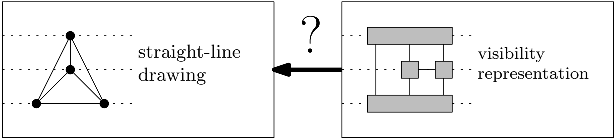

Theorem [B. 2012]: Every planar flat visibility representation of height h can be converted into a planar straight-line drawing of height h.

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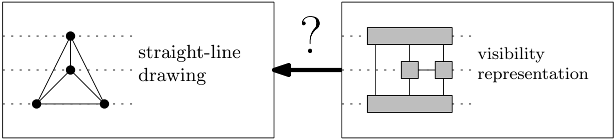

Theorem [B. 2012]: Every planar flat visibility representation of height h can be converted into a ~~planar~~ straight-line drawing of height h.

INCORRECT PROOF!

Graph drawing styles

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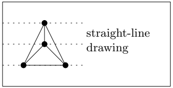

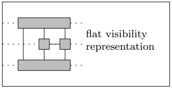

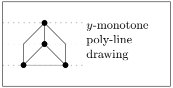

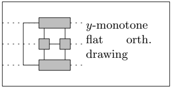

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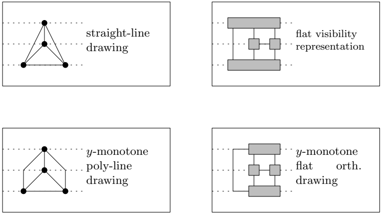

- *Flat* drawing: Vertices are horizontal segments (or points).

Graph drawing styles

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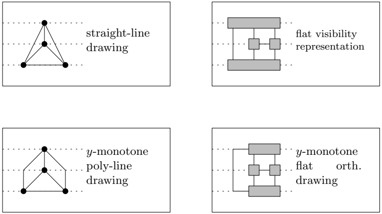
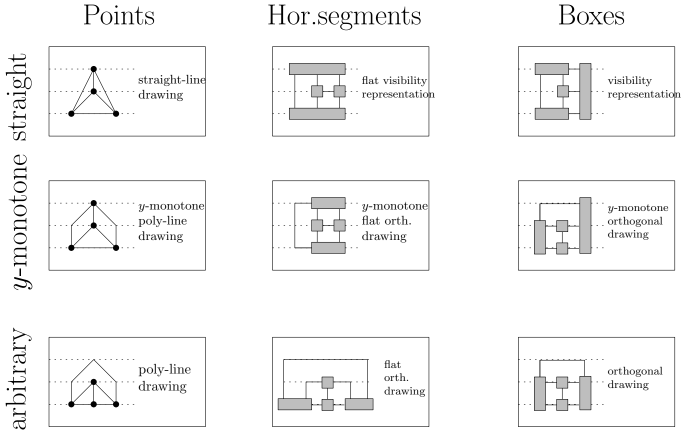


- *Flat* drawing: Vertices are horizontal segments (or points).
- *y-monotone* drawing: Edge curves have no local minima/maxima.

Height-preserving transformations



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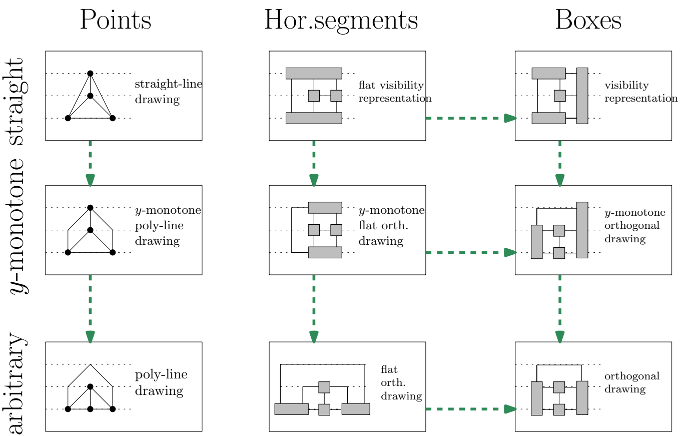
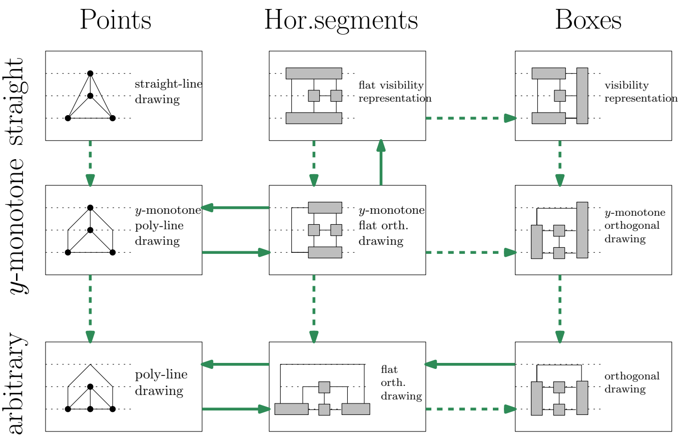


Trivial transformations:

E.g. straight-line drawings **are** y-monotone poly-line drawings.

Height-preserving transformations

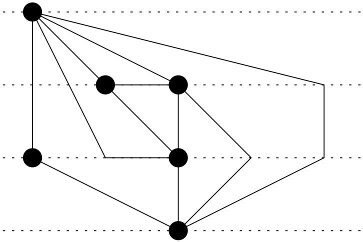


Easy height-preserving transformation:

E.g. poly-line drawing → flat orthogonal drawing.

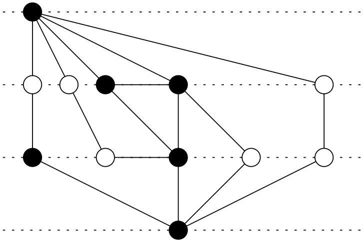
Proof by example

Poly-line drawing → flat orthogonal drawing



Proof by example

Poly-line drawing $\rightarrow$ flat orthogonal drawing



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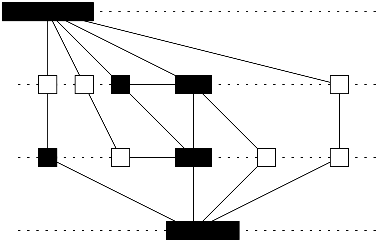

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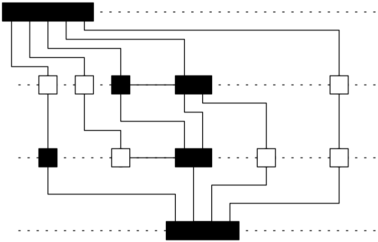

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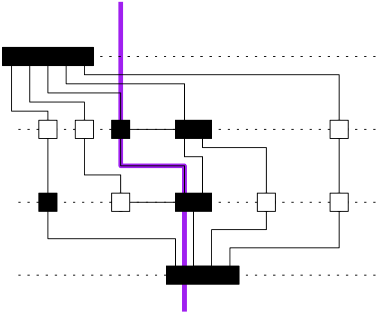

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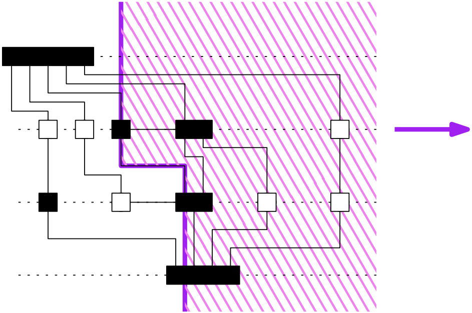

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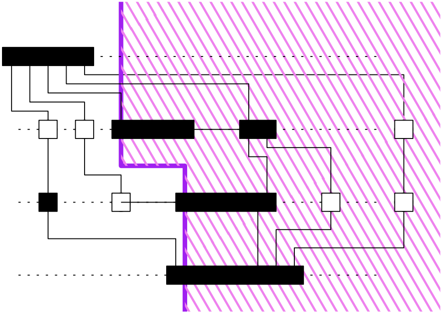

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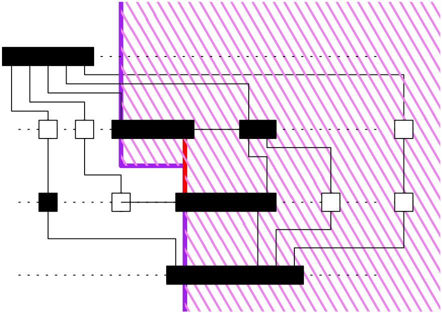

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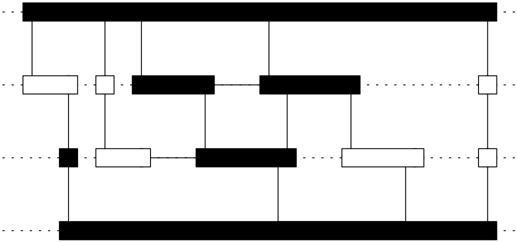

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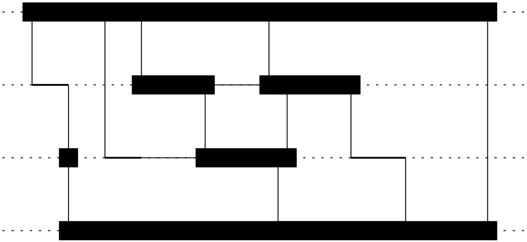
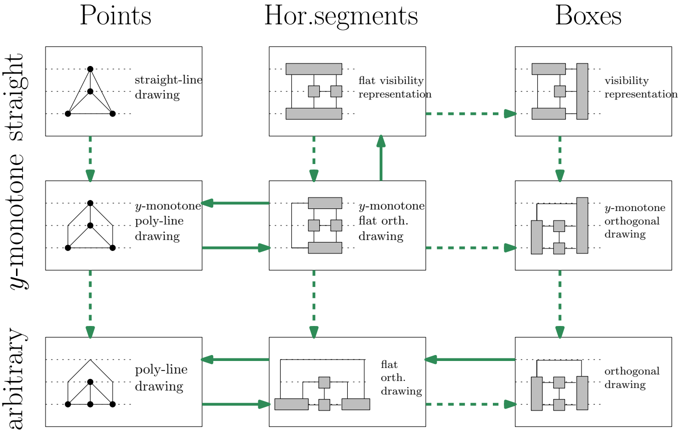


Height-preserving transformations



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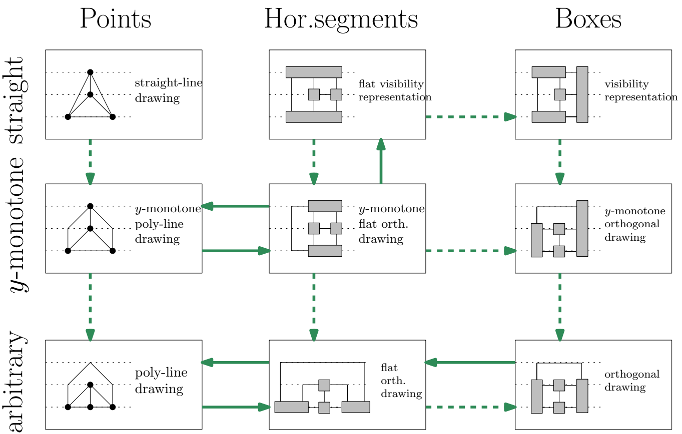
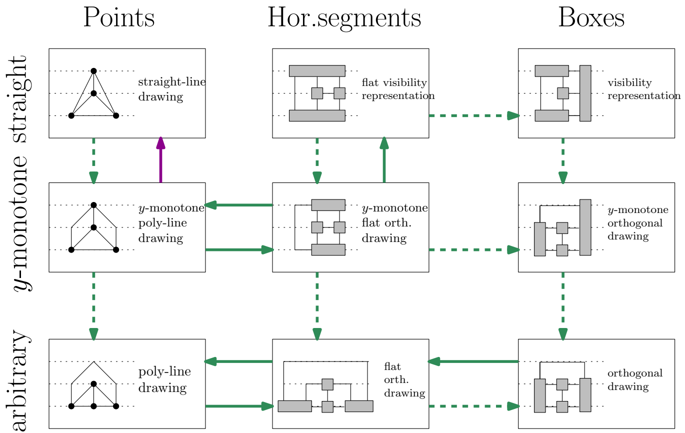


Not easy: Creating straight-line drawings.

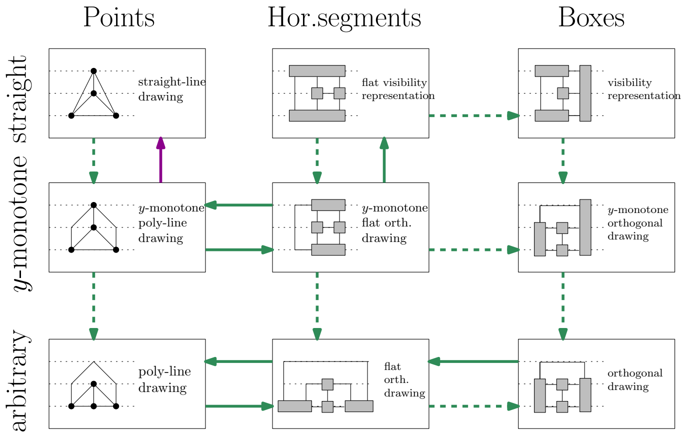
Height-preserving transformations



Theorem (B., 2014)

Every planar y-monotone poly-line drawing can be converted into a straight-line drawing of the same height.

Height-preserving transformations



Theorem (~~B., 2014~~ Pach, Tóth, 2004)

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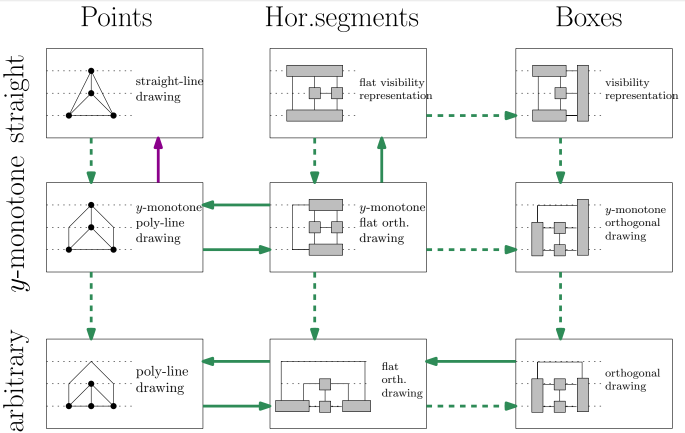
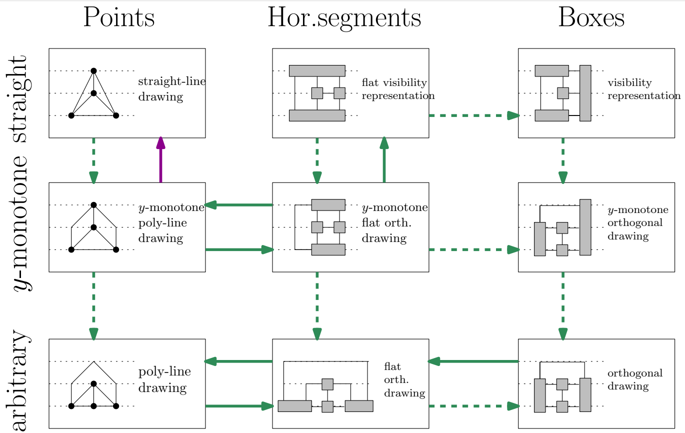


Theorem (Pach, Tóth, 2004)

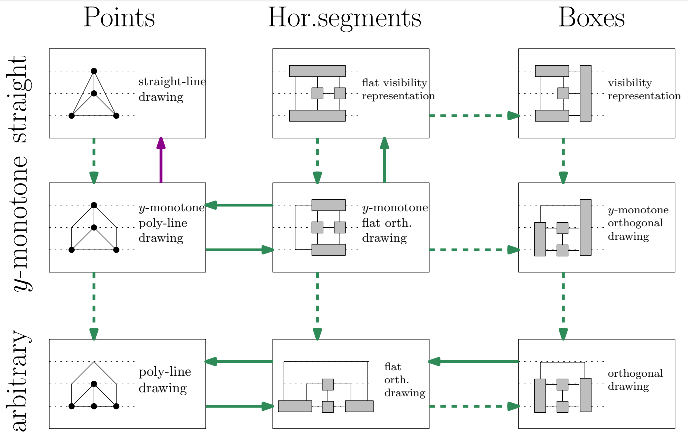
Every planar strictly y-monotone poly-line drawing can be converted into a planar straight-line drawing of the same height but it may take exponential time.



Theorem (~~Pach, Tóth, 2004~~ Eades et al. 1996)

Every planar strictly y-monotone poly-line drawing can be converted into a planar straight-line drawing of the same height and it ~~may take exponential time~~ can be done in linear time.

Height-preserving transformations



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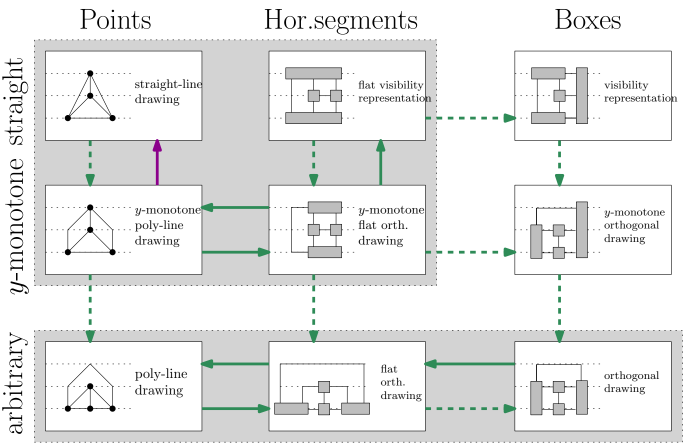
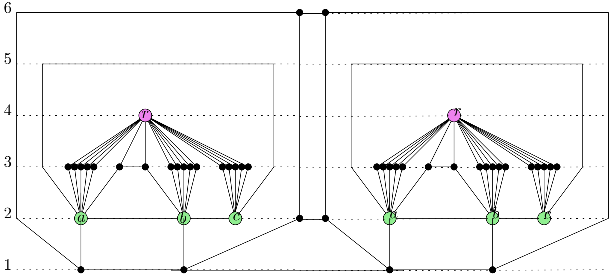


Get equivalence classes:

All styles within class are equally good (with respect to height).

But what about other pairs?

A nasty example



- Has a poly-line drawing on 6 rows.
- Consider any orthogonal drawing on 6 rows.

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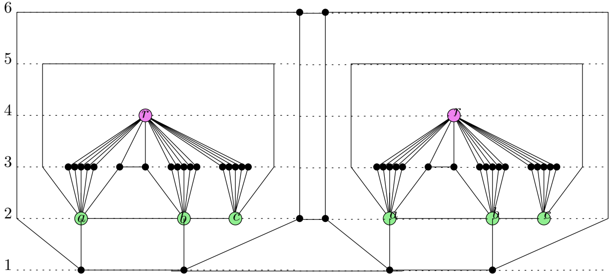

- Has a poly-line drawing on 6 rows.
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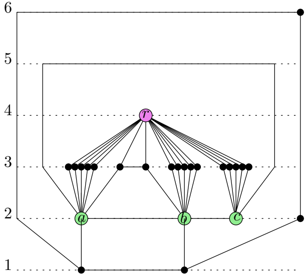
- Must use this planar embedding (essentially 3-connected.)
- One subgraph must use this outer-face.

A nasty example



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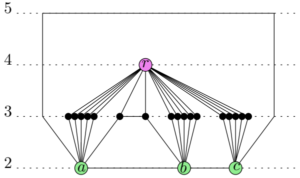

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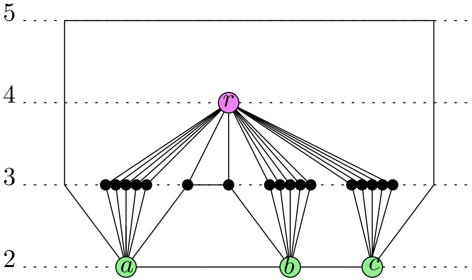

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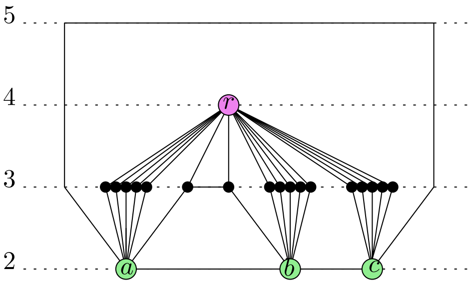

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- Inside drawn within rows $2-5$
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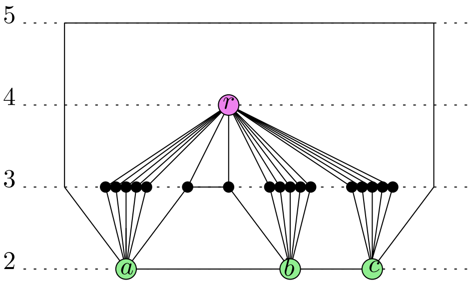

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$\Rightarrow$ some row separates r from a, b, c.

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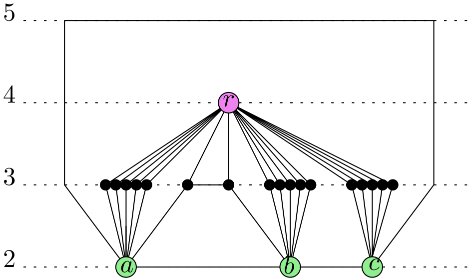

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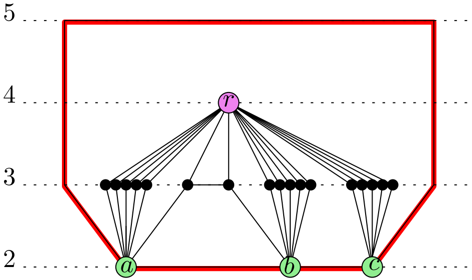

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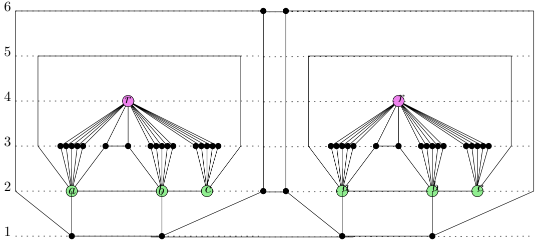

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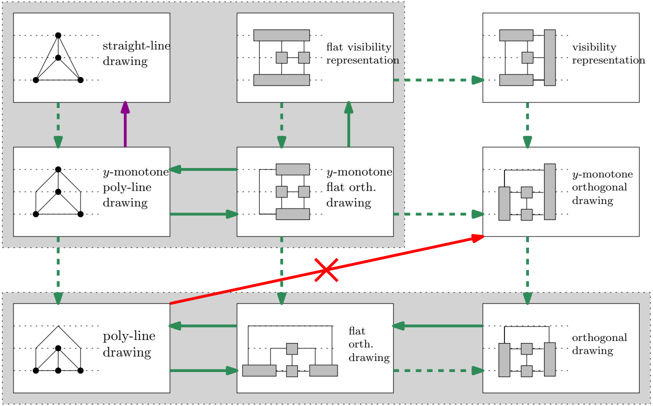
$\Rightarrow$ boxes for a, b, c are segments in the **same** row.

$\Rightarrow$ triangle $\{a, b, c\}$ is *not* drawn y-monotonically.

No y-monotone orthogonal drawing on 6 rows!

Height-preserving transformations



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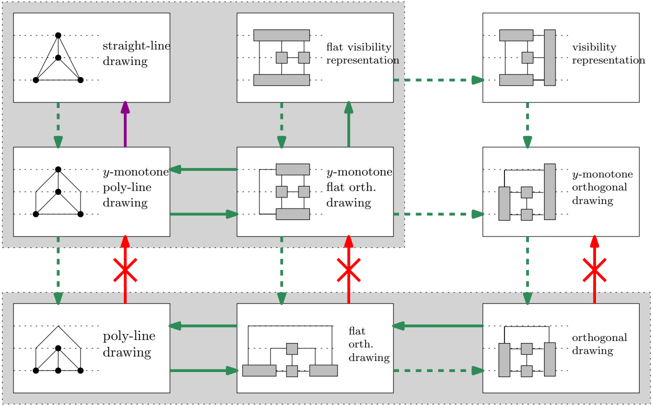

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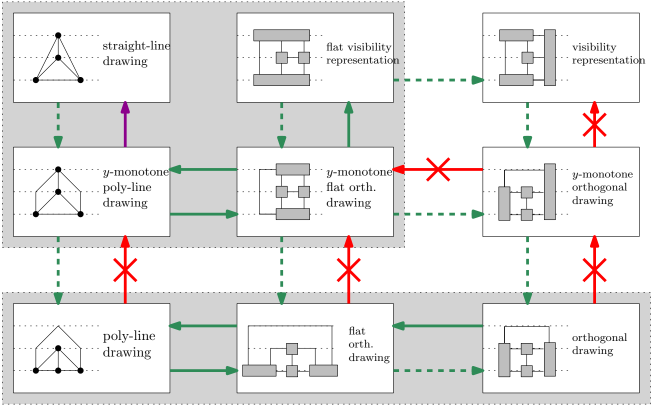

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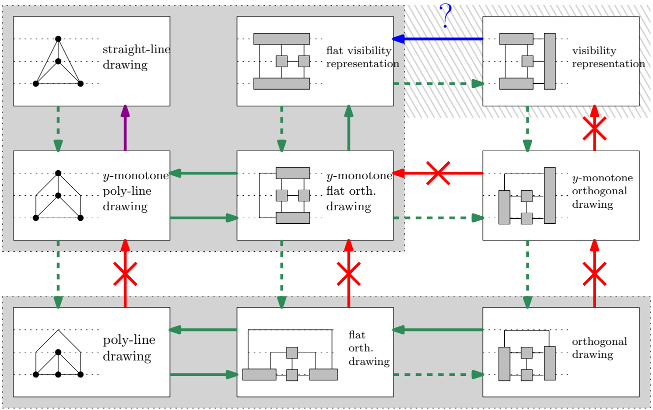

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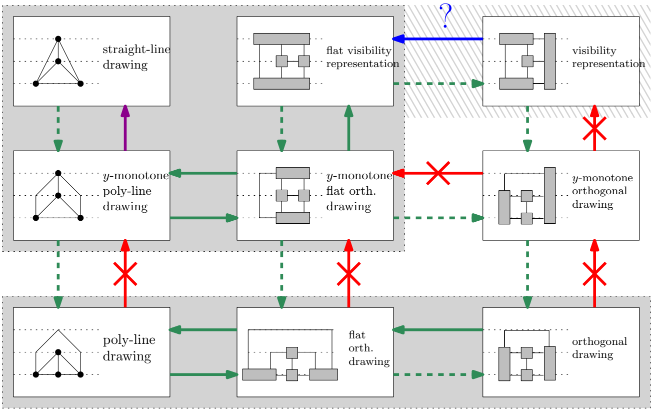


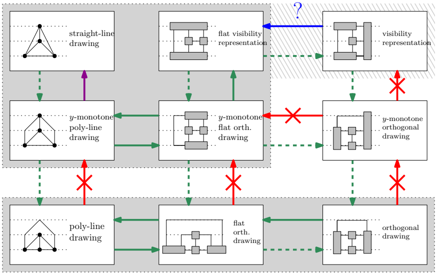
Conjecture

Every visibility representation (even non-flat) can be converted into a straight-line drawing of the same height.

(True for bipartite graphs.)

Width considerations

But: want to preserve **area**. How does the width change?



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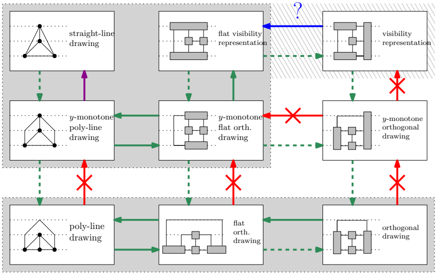

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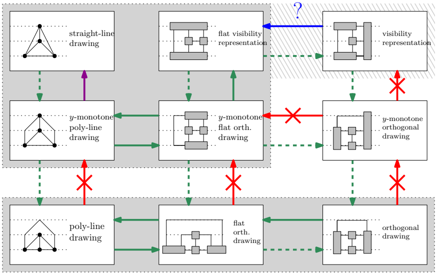

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- Visibility representations: width $\leq \max\{m, n\}$.

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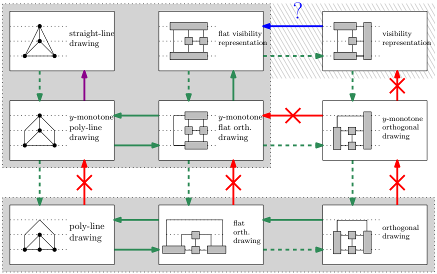

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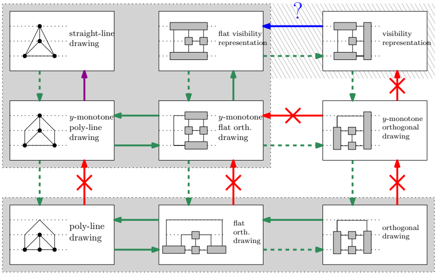

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- Straight-line drawings: ???

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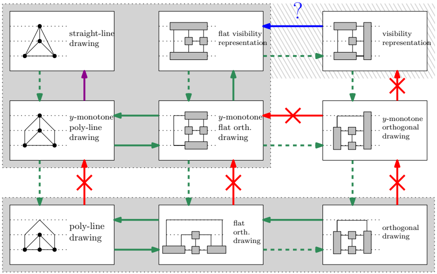

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- Straight-line drawings: ???
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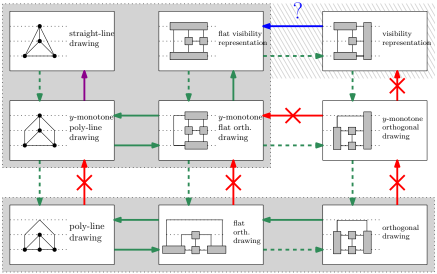
Width considerations

But: want to preserve **area**. How does the width change?



- Creating poly-line drawings: width does not increase.
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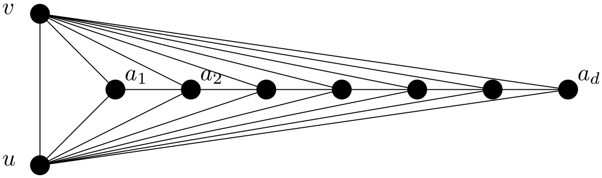
  - Eades & Lin 1997: $\Omega(n(h-1)!)$ width required for fixed y-coordinates.

Another bad example



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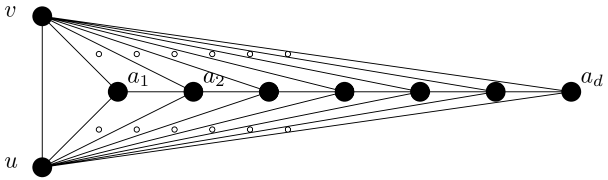

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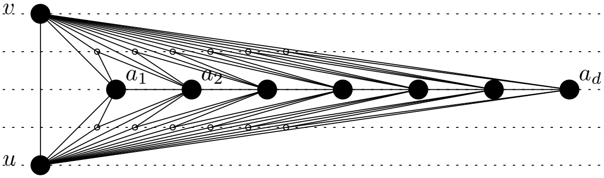

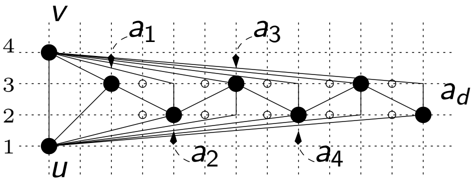

- Has a y-monotone poly-line drawing on 4 rows

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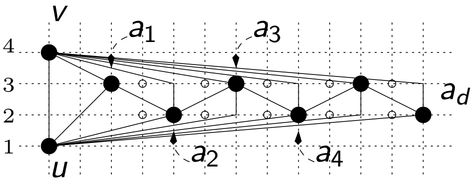

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$\Rightarrow$ Has a straight-line drawing on 4 rows. Fix one.

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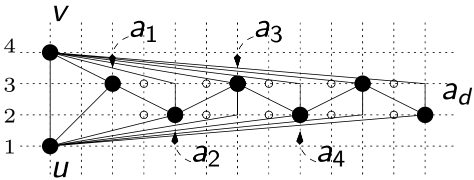

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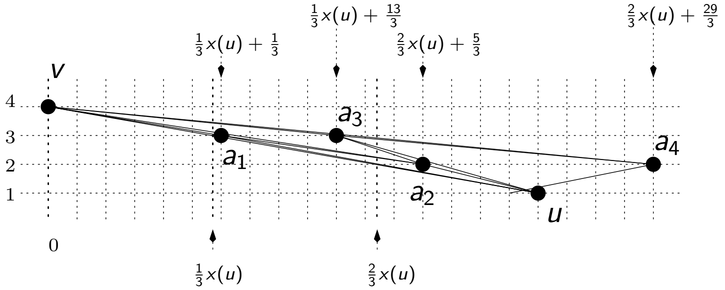

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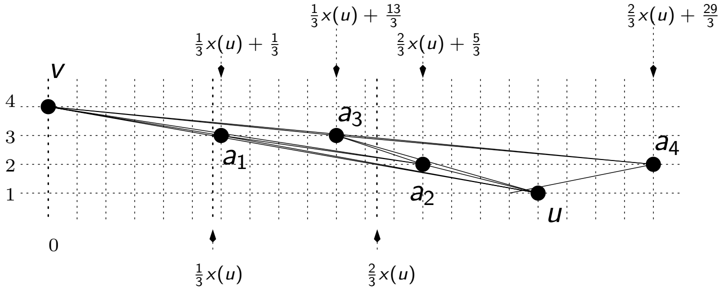

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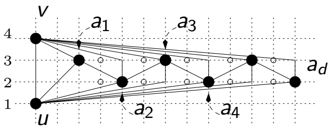
⇒ Width is $\geq 2^{n/3}$—exponential!

Width of straight-line drawings



Theorem

There exists a planar graph that has a straight-line drawing on 4 rows, but any such drawing has width $\Omega(2^{n/3})$.

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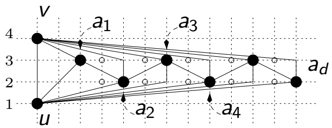

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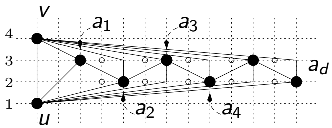

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Corollary

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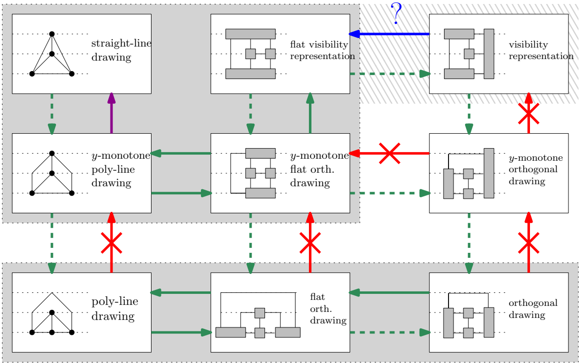
Conjecture

There exists a planar graph that has a straight-line drawing on h rows, but any such drawing has width $\Omega(h^{\theta(n)})$.

Conclusion and Open Problems

Many height-preserving transformations of planar drawings:



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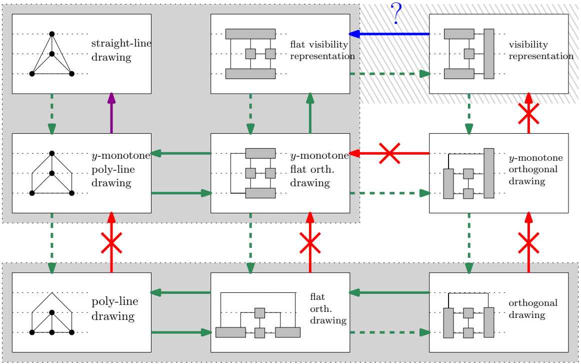

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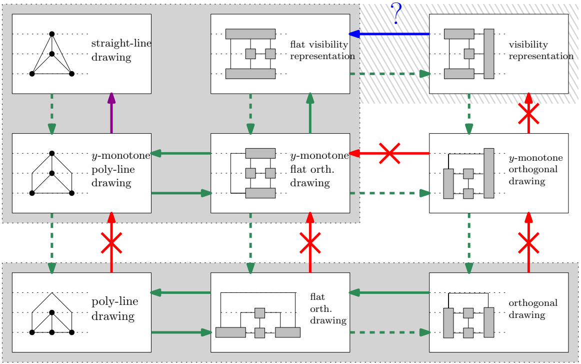

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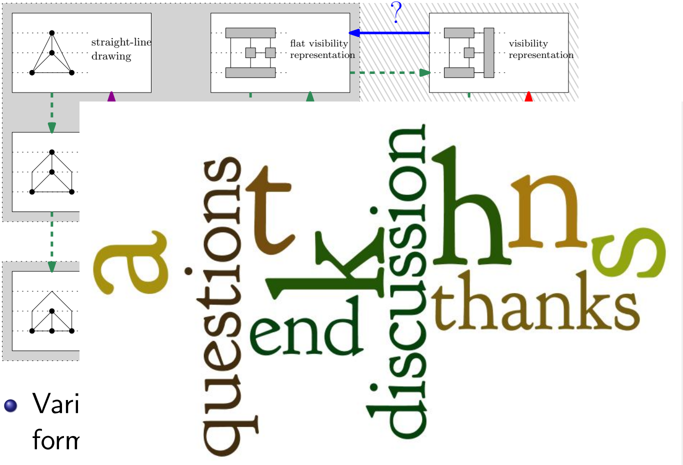

- Vari … ngs, IP form
- Transformations of non-planar drawings? (1-planar? Fan-planar?)